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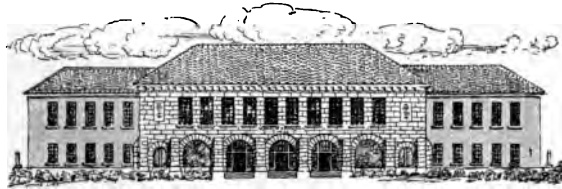


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COURSE OF MATHEMATICS

ELEMENTS OF GEOMETRY

JOHN MACNIE

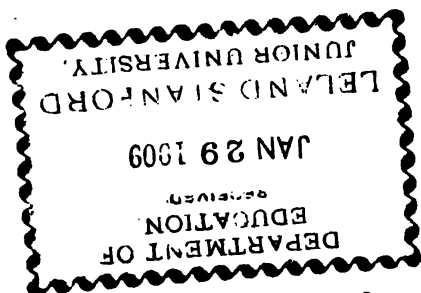


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WHITE'S SERIES OF MATHEMATICS

ELEMENTS OF GEOMETRY

PLANE AND SOLID

BY

JOHN MACNIE, A.M.

AUTHOR OF "THEORY OF EQUATIONS"

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AUTHOR OF WHITE'S SERIES OF MATHEMATICS



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FIRST BOOK OF ARITHMETIC.

NEW COMPLETE ARITHMETIC.

SCHOOL ALGEBRA. (*In Preparation.*)

ELEMENTS OF GEOMETRY.

ELEMENTS OF TRIGONOMETRY. (*In Preparation.*)

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PREFACE.

IN this treatise, an endeavor is made to present the elements of geometry with a logical strictness approaching that of Euclid, while taking advantage of such improvements in arrangement and notation as are suggested by modern experience. It has been carefully kept in mind that the purpose of such a work is only in a secondary degree the presentation of a system of useful knowledge. A much more important purpose is to afford those who study this subject the only course of strict reasoning with which the great majority of them will ever become closely acquainted. A mind that, by exercise in following and weighing examples of strict logical deduction, has learned to appreciate sound reasoning, and, by practice on suitable exercises, has been trained to reason out a sound logical deduction for itself, has gained what is of far greater importance than mere knowledge; it has gained power. A treatise on rational geometry ought, accordingly, to have for guiding principles those laid down by Pascal as the chief laws of demonstration, substantially as follows: to leave no obscure terms undefined; to assume nothing not perfectly evident; to prove everything at all doubtful, by reference to admitted principles.

In accordance with the first principle, great care has been taken in the wording of the definitions. In the case of some terms, such as *straight line* and *angle*, for which no definitions quite free from objection have as yet been proposed, those adopted have been chosen, not as theoretically perfect, but as best suited to the comprehension of the beginner, and most available in deducing the properties of the things defined.

The use of hypothetical constructions has been abandoned for several reasons. To assume them silently, as is now usually done, is unwarrantable in a treatise upon a science supposed, above all others, to consist of a series of rigorous deductions from admitted truths. Why state so carefully that we must *assume* the possibility

of drawing or prolonging a straight line, and say nothing in regard to constructions so much less obvious? The author's experience in teaching geometry has convinced him that these stealthy assumptions are decidedly adverse to the acquisition, on the part of the learner, of those habits of strict reasoning which it is the main object of geometrical study to impart. The teacher should have it in his power to inquire, at every step, not only why such a statement is true, but also why such a construction is allowable. On the other hand, the insertion of the few problems really needed as auxiliaries in the demonstration of theorems, imparts to their sequence a logical consistency that cannot obtain where the learner is continually required to perform operations, the possibility of which has neither been proved nor formally assumed.

In regard to the use of circumferences as construction lines in the first book, it may be remarked that, of all lines, the circumference is the easiest to define, to construct, and to conceive, — far more so than is the case with the straight line. No property of the circle, not even its name, is introduced in the first book, but only that property of the circumference which is given in its definition, and some immediate consequences from that property. The employment, at the outset, of the simpler of the two lines treated of in elementary geometry would hardly require apology or explanation but for the force of custom. Yet, strangely enough, in treatises that scrupulously defer the definition of the terms *circumference* and *radius* to a subsequent book, there seems to be no scruple against the employment of arcs in illustration of the nature of angles, and we see gravely laid down the postulate: *A circumference may be described about any point as center, etc.*

The deviation in this work from the usual order of propositions is comparatively slight. In the early part of the first book, so important as the foundation of the science, the properties of triangles are introduced immediately after the discussion of the general properties of angles. This arrangement, especially as regards the different cases of equal triangles, presents several advantages: these propositions are immediately deducible from first principles, or from each other; they are easily grasped by the beginner; above all, they are of the highest utility as aids to further acquisition. It is in itself no slight advantage for the learner to become accustomed, from the first, to the use of these important auxiliaries in demon-

stration. From the second book, again, certain propositions that treat of proportional angles have been removed to the place where they belong, after the discussion of ratio and proportion. In the treatment of these subjects, while adhering to the now prevalent method, an endeavor has been made to obviate one frequent source of confusion by making a clear distinction between concrete quantities and their numerical measures.

In regard to propositions and corollaries, the rule observed has been to admit only such as are important in themselves, or have a bearing on subsequent demonstrations and studies. In this era of over-crowded curricula, the aim of an elementary text-book should be to present the necessary rather than the novel or merely interesting. For this reason some subjects have been relegated to an appendix, where they may be studied or omitted according to circumstances.*

The exercises have been carefully selected with a view to their bearing upon important principles, and are, with few exceptions, of such slight difficulty as not to discourage the learner of average ability. In the first sets of exercises, ample assistance is afforded the student by means of references and diagrams, — aids that are withheld from the point where the student should have learned to help himself. It is by no means expected that the average class will find time for all the exercises, but enough are given to afford the teacher full opportunity of choice.

SUGGESTIONS. It is earnestly recommended that, before any book work is assigned to a beginning class, a lesson be devoted to the constructions given in Arts. 200–206. The teacher, having shown on the blackboard, for example, how to bisect a straight line, should set the class to doing the same, and require each pupil to bring in, next day, one or more neatly worked examples of the required constructions. The practical familiarity thus gained with the geometrical concepts involved will amply repay the time thus spent. The two great sources of difficulty to the beginner in geometry are the comparative novelty of the subject matter and the unaccustomed clearness of conception and exactness of expres-

* The whole of the last part of Book IX., treating of spherical angles and polygons, may as well be omitted by pupils not to take up spherical trigonometry.

sion required in this new study; it will be found that the second source of difficulty is most easily diminished by reducing the first to a minimum.

The easy exercises at the foot of the page can be employed to the best advantage as material for impromptu work, the teacher giving as much aid, by suggestive questions, as will enable the class to solve them offhand. Such as prove too difficult for this first attempt may then be assigned as work to be prepared. The questions found at the end of most books may either be taken up as the class progresses through the book, or be left for review. Such matter as that found on p. 105 is to be carefully read over in the class with a running comment by the teacher; the gist of the ideas, not the words in which they are expressed, being what the pupil should try to retain.

The pupil should be required to give the exact words of definitions, axioms, postulates, and enunciations. In other matters some latitude may be allowed, though occasion should be taken to point out in what respects the pupil's own wording may be objectionable.

Outside of Euclid, it is of doubtful utility to exact from pupils a knowledge of all propositions by number. The axioms, postulates, and certain important propositions should be known by number, but in written or oral work, other references may be given by abbreviated quotations.

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SYMBOLS.

\therefore <i>because or since.</i>	\angle <i>angle.</i>
\therefore <i>therefore.</i>	st. \angle <i>straight angle.</i>
$+$ <i>plus.</i>	rt. \angle <i>right angle.</i>
$-$ <i>minus.</i>	\triangle <i>triangle.</i>
$=$ <i>is equal to.</i>	rt. \triangle <i>right triangle.</i>
\approx <i>is equivalent to.</i>	AB , <i>line or side AB.</i>
$>$ <i>is greater than.</i>	comp. <i>complement.</i>
$<$ <i>is less than.</i>	supp. <i>supplement.</i>
\neq <i>coincides with.</i>	resp. <i>respectively.</i>
\parallel <i>parallel.*</i>	REQ. <i>required.</i>
\perp <i>perpendicular.*</i>	Q.E.D. <i>as was to be proved.</i>
\odot <i>circle.</i>	Q.E.F. <i>as was to be done.</i>

The symbols marked above with an asterisk are to be read as nouns when preceded by an article or similar word. Plurals are formed by adding 's to the singular. Thus, \triangle 's, \parallel 's, etc.

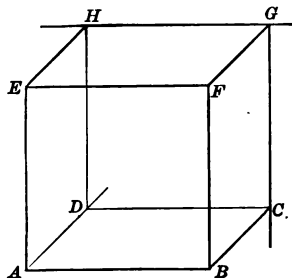
In reading such expressions as $\triangle A'B'C'$, $\odot abcd$, it will be sufficient, in almost every case, to give the distinguishing epithet to the last letter only. Thus, the above expressions may be read respectively: *triangle ABC prime, circle abcd small or minor.* It is, of course, seldom necessary to say the *line* or the *side AB*; it is usually sufficient to say *AB*.

PART I. PLANE GEOMETRY.



INTRODUCTION.

WE conceive space as extending without limit in every direction. Each physical body, a marble cube, for example, occupies a limited portion of space. Thus the cube AG is limited on all sides by boundaries which are called *faces*; as AF , EG , etc. These faces, again, meet in edges, which are called *lines*; as AB , FG , etc.; and finally, these lines meet in extremities, which are called *points*; as A , F , H , etc. If, now, leaving its material entirely out of consideration, we think only of the portion of space occupied by the cube, with its faces, edges, and points, we have before us a so-called *geometrical solid*. It is with such abstract solids as this that geometry is concerned.



DEFINITIONS.

1. A *solid* is a limited portion of space, and has three dimensions: *length*, *breadth*, and *thickness*.
2. *Surfaces* are the limiting boundaries of solids, and have two dimensions: *length* and *breadth*.

3. *Lines* are the boundaries or intersections of surfaces, and have but one dimension: *length*.

4. *Points* are the extremities or intersections of lines; hence a point has no dimension, but only *position*.

Just as we conceive a geometrical solid apart from material, so we may conceive of surfaces apart from solids, of lines apart from surfaces, and of points apart from lines. These abstract points, lines, surfaces, and solids are the so-called *space-concepts*, the elements of our notions of space.

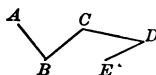
5. *Geometry* is the science that treats of the properties and relations of space-concepts, also called *geometrical concepts*.

Each geometrical concept has *position*, determined by the location of its points in space; and all except points have *form* or *shape*, determined by the relative position of their points, and *extent* or *magnitude*, determined by the nearness or remoteness of their bounding points.

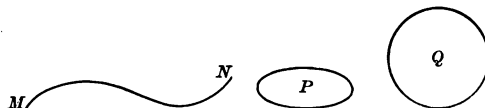
6. A *straight line* is a line that does not change its direction at any point; as AB .



7. A *broken line*, as ABC or $ABCDE$, changes direction at one or more points.



8. A *curved line*, as MN , changes direction at every point; *i.e.*, no part of it is a straight line. The curve is said to be *closed* when it forms a continuous boundary. Thus P and Q are closed curves.



The term *line*, when used hereafter by itself, is to be understood as denoting a *straight line*; similarly, *curve* will signify *curved line*.

9. A *plane surface* or *plane* is such that a straight line passing through any two points in that surface lies wholly in the surface.

10. A *geometrical figure* is any given combination of points, lines, or surfaces; the representation of a figure is a *diagram*.

Just as a diagram is the representation of a figure, so the lines of the diagram and the surface on which it is drawn are merely more or less imperfect representations of the ideally perfect lines and planes to be treated of.

11. A *plane figure* is such that all its points lie in the same plane.

12. *Plane geometry* is the geometry of plane figures.

13. A *magnitude* is a concept any part of which is of the same kind as the whole.

Thus since any part of a line is a line; any part of a surface, a surface; any part of a solid, a solid, — lines, surfaces, solids, and angles, presently to be defined, are *geometrical magnitudes*. Other magnitudes are *time, weight, mass*, etc.

14. *Equal magnitudes* are such as can be made to coincide exactly.

Magnitudes coincide exactly when, one being placed upon the other, every point of the one lies upon a corresponding point of the other.

LINES.

From the definition of the straight line (6),* it follows: —

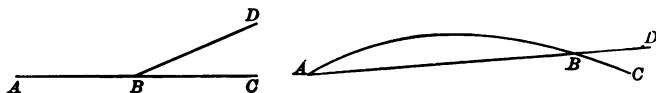
15. *Straight lines that coincide in part coincide throughout.*

16. *Two points determine a straight line.*

17. *Two straight lines cannot inclose a surface.*

* Numbers in parentheses throughout the book refer in general to paragraphs.

For if two lines ABC , ABD , coincide in a part AB only, or in two points A and B only, they cannot both be straight lines (6), since one, at least, must change direction.



18. Since the direction of one point from another may be regarded as the path of a point that passes along the straight line that connects these points, and as, if A and B are the points, the direction may evidently be either from A to B or from B to A , the same straight line may mark either of two opposite directions, or it may be regarded as extending in both directions from any point in it.

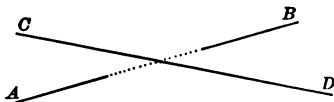


19. A line is said to be *indefinitely produced* when prolonged as far as necessary, or without assigned limit. If two straight lines AB , CD , lie in the same plane, it is evident that, if *indefinitely produced*, they must either meet or not meet. Thus:—

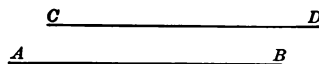
(a) If they can be produced so as to have *more than one* common point, they lie in the same line and have the *same* directions.



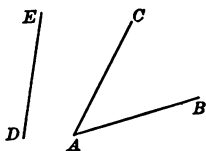
(b) If they can have *but one* common point, they can *intersect* in that point, and they have *different* directions.



(c) If they can have *no* common point, — that is, cannot meet, — they are said to be *parallel*.

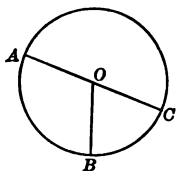


20. If the straight lines drawn from A to B and from D to E are equal, the *distance* from A to B is said to be *equal* to that from D to E . If the distances from A to B and C are equal, A is said to be *equidistant* from B and C , while B and C are said to be *equally distant* from A .



21. A *circumference* is a closed curve described in a plane, and is such that all its points are equally distant from a point within the curve, called the *center*.

22. A *radius* is any straight line drawn from center to circumference. Thus OA , OB , OC , are radii of the circumference ACB , whose center is O .

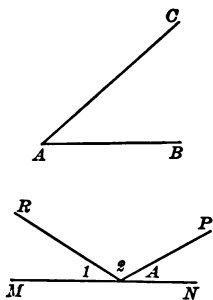


23. All radii to the same circumference are equal. It is also obvious that a point in its plane lies *within* or *without* a given circumference, according as its distance from the center is *less* or *greater* than the radius.

ANGLES.

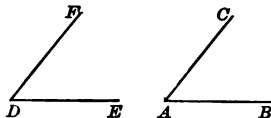
24. A *plane angle* is the opening or difference of direction between two lines that meet or might meet. The point of meeting is called the *vertex*; and the lines, the *sides* of the angle.

Thus the lines AB , AC , meeting in the vertex A , are said to *form* or *include* the angle called BAC , CAB , or simply the angle A when no other angle has the same vertex. Even when several angles have a common vertex, it is sometimes convenient to designate each by a number or letter placed within, near the vertex. Thus the angles MAR , RAP , PAN , may be more briefly designated as the angles 1, 2, and A , respectively.



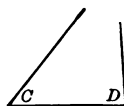
25. Angles are *equal* if their sides can be made to coincide.

Thus the angles BAC and EDF are equal if, DE being made to lie on AB , DF can also be made to lie on AC , which is possible only when the opening between DE and DF is the same as the opening between AB and AC .

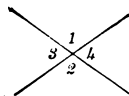


26. *Adjacent angles* are such as have a common vertex and a common side that separates them. Thus the angles 1 and 2 in Art. 24, or 2 and 4, are adjacent angles.

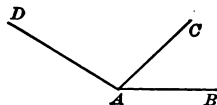
Angles that have a common side, as D and C , but separate vertices, are sometimes called adjacent. It is better, however, to avoid possible confusion by calling them *including angles*, as including the common side between their vertices.



27. *Vertical angles* are the opposite angles formed by the intersection of two straight lines. Thus the angles 1 and 2 are vertical angles, as are also the angles 3 and 4.



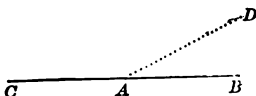
28. The *sum* of two angles is equal to the angle formed by placing them so as to be adjacent. The *difference* of two angles is the angle that added to the smaller angle makes an angle equal to the greater.



Thus BAD is the sum of BAC and CAD ; BAC is the difference of BAD and CAD .

29. A *straight angle* is one whose sides lie in opposite directions from the vertex, so as to be in a straight line.

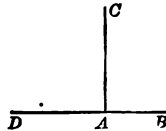
Thus BAC is a straight angle if its sides AB , AC are in a straight line. It may be regarded as formed by two lines drawn in opposite directions from A ; or by opening out a smaller angle, BAD , until its sides lie in opposite directions; or as the



sum of two adjacent angles, BAD , DAC , whose exterior sides lie in a straight line.

30. When the adjacent angles formed by one straight line meeting another are equal, each angle is called a *right angle*.

Thus, if the angles BAC , CAD , formed by CA meeting BD in A , are equal, each of them is a right angle.

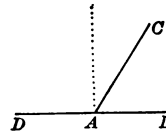


31. A *right angle* is equal to one half of a straight angle.

This follows from the preceding definitions; for the two equal angles, CAB , CAD , make up the straight angle BAD .

32. An *acute angle* is less than a right angle; an *obtuse angle* is greater than a right angle and less than a straight angle. Thus BAC is an acute, and CAD an obtuse, angle.

Acute angles and obtuse angles are called *oblique angles*; and lines are said to be *perpendicular* or *oblique* to each other according as they meet at a right or an oblique angle.



33. If the sum of two angles is equal to a right angle, each is called the *complement* of the other, and the angles are said to be *complementary*. If the sum of two angles is equal to a straight angle, each is called the *supplement* of the other, and the angles are said to be *supplementary*.

QUESTIONS.

1. In a line AB , take any point C ; then AB is the sum of what two lines? AC is the difference of what two?



2. If you fold a piece of note paper so as to form an edge, what sort of a line is formed?

3. If you fold the paper again, so as to double that edge upon itself, what angle will the second edge thus formed make with the first?

4. If you suspend a weight by a string, in what sort of a line is the string stretched ?

5. If you whirl the weight round at the end of the string, in what sort of a line does the weight move ?

6. What sort of a surface is presented, roughly speaking, by the walls of a room ? By the surface of a floor ? By the surface of a slate ? Mention other like surfaces.

7. How would you apply a straightedge or ruler so as to ascertain whether a given surface is a plane ? What property of planes do you apply (Art. 9) ?

8. Straight lines can be drawn on the surface of a stovepipe, and yet it is not a plane : why not ?

9. Can a straight line be drawn on the surface of an eggshell ? If not, what kind of a line can be drawn on such a surface ?

10. From a point O draw lines OA , OB , OC , OD , in one plane. Name each of the angles thus formed. Which are adjacent angles ? Name one that is the sum of two ; of three.

11. Can you draw two angles that have a common vertex and a common side, and yet are not adjacent ?

12. What sort of an angle is less than its supplement ? Is equal to its supplement ? Is greater than its supplement ?

PROPOSITIONS.

The truths of geometry are presented for consideration under the form of general statements called *propositions*.

34. A *theorem* is a proposition stating a geometrical truth.

35. A *problem* is a proposition stating a proposed construction.

36. A *corollary* is a theorem that follows so plainly as a consequence from a preceding proposition, or definition, that its formal proof is either omitted or is merely indicated.

Thus in Arts. 15, 16, 17, are given certain important corollaries from the definition of the straight line ; and in Art. 23 we have a very obvious corollary from the definitions of circumference and radius.

- 37. An *axiom* is a theorem assumed as self-evident.
- 38. A *postulate* is a problem assumed as possible.
- 39. A *scholium* is a remark upon a preceding proposition.

40. The axioms and postulates, together with the definitions, constitute the logical basis of geometry. Axioms express certain simple truths in regard to magnitude in general, — truths so confirmed by all our experience that the mind cannot conceive their opposites as true. All the axioms except the last two, which are really definitions of the terms *whole* and *part*, might be deduced from the first. They are such obvious truths, however, that it is deemed sufficient to state them for convenience of reference.

AXIOMS.

- 1. *Magnitudes equal to the same or equal magnitudes are equal to each other.*
- 2. *If equals are added to equals, the sums are equal.*
- 3. *If equals are taken from equals, the remainders are equal.*
- 4. *If equals are added to unequals, the sums are unequal.*
- 5. *If equals are taken from unequals, or unequals from equals, the remainders are unequal.*
- 6. *The doubles of equals are equal.*
- 7. *The halves of equals are equal.*
- 8. *The whole is greater than any of its parts.*
- 9. *The whole is equal to the sum of all its parts.*

POSTULATES.

Let it be assumed that, in a given plane,

- 1. *A straight line can be drawn joining any two given points.*
- 2. *A given straight line can be produced to any extent.*

3. *On the greater of two straight lines a part can be laid off equal to the less.*

4. *A circumference can be described from any center, with any radius.*

5. *A figure can be moved unaltered to a new position.*

It is assumed in these postulates that we have at our disposal, (1) a plane extending indefinitely in all directions, (2) some means of causing a marking point to move in a straight line in any given direction, (3) some means of causing a marking point to move in that plane so as always to remain at a given distance from a given point in it. The plane may be represented by a blackboard or flat piece of paper; the second requirement is met by the use of a straightedge and marking point; the third, by a pair of compasses, or other device. By means of Post. 5, containing the principle of *superposition*, we are enabled to apply the criterion of Art. 14 in order to prove the equality of two given magnitudes. Thus two straight lines would be proved equal by placing them so as to coincide end with end; two angles, by causing their sides to coincide; and so on.

METHOD OF PROOF.*

In general, the statement and proof of a proposition consist of several distinct parts; the *enunciation*, the *construction*, and the *demonstration*.

1. The GENERAL ENUNCIATION or statement consists of an *hypothesis* (or supposition) and a *conclusion*. Thus in Prop. I. we have, though in different words:—

HYPOTHESIS. *If a line is perpendicular to a second line at a certain point,*

* To be read in connection with Prop. I.

CONCLUSION. *No other line in the same plane can be perpendicular to the second line at that point.*

2. The PARTICULAR ENUNCIATION, again, refers us to a particular figure or figures fulfilling the given conditions. The hypothesis and conclusion of the particular enunciation will be distinguished by the headings *Given*, and *To Prove*, respectively.

3. In the CONSTRUCTION we apply the postulates, or problems that have been proved possible, to make such changes in the form or position of the given figures as may be needful for the demonstration.

4. In the DEMONSTRATION we deduce, by a train of reasoning, the proposition to be proved, from other propositions already proved or granted. Thus, in Prop. I., we show, by means of Ax. 8 and Ax. 1 and the definition of right angles, that the angles formed by the line AE with BD must be unequal, and therefore cannot be right angles.

In problems, the construction, not always a necessity in the proof of a theorem, is the essential part. Instead of the heading *To Prove*, however, we put the heading *Required*, as indicating what is required to be done.

QUESTIONS.

1. What kind of a surface and what instruments are assumed as necessary in the constructions of plane geometry?

2. How do you draw a straight line longer than your straightedge? What property of straight lines do you apply?

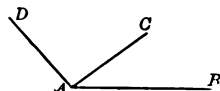
3. With what instrument do you lay off on a line AB a shorter line CD ?

4. Draw a line equal to the given line AB ; produce it to E , so that BE shall be equal to the line CD : AE is the sum, and BE the difference, of what lines?

5. The sum of a right angle and an acute angle is necessarily what sort of an angle?

6. The difference of an obtuse angle and a right angle is necessarily what sort of an angle?

7. In the annexed diagram, show that there are three different angles having the same vertex A . Name them.



8. If BA were produced through A to E , how many different angles would have the same vertex A ? Name them.

Read the following expressions:

$$9. \angle BAC + \angle CAD = \angle BAD.$$

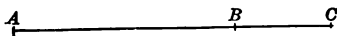
$$11. \angle BAD > \angle CAD.$$

$$10. \angle BAD - \angle BAC = \angle CAD.$$

$$12. \angle BAC < \angle BAD.$$

Express in symbols each of the following statements:

13. The sum of the lines AB and BC is equal to the line AC .

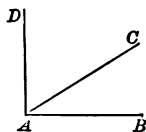


14. The difference of the lines AC and BC is equal to the line AB .

15. The line AC is greater than the line BC .

16. The line AB is less than the line AC .

17. The sum of the angles BAD and BAC is a right angle.



18. The difference of the angles BAD and BAC is equal to the angle CAD .

BOOK I.

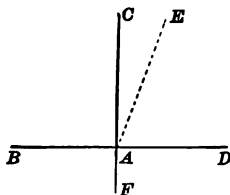
LINES AND RECTILINEAR FIGURES.



ANGLES.

PROPOSITION I. THEOREM.

41. *At a given point in a straight line there can be in the same plane but one perpendicular to that line.*



Given: CF perpendicular to BD at A ;

To Prove: No other line drawn through A can be \perp to BD .

For any other line drawn from A must lie between FC and AB or between FC and AD . Let it lie between FC and AD , as AE .

Because $\angle CAB = \angle CAD$, (Hyp.)

but $\angle EAB > \angle CAB$, and $\angle EAD < \angle CAD$, (Ax. 8)

$\angle EAD$ is not equal to $\angle EAB$, (Ax. 1)

$\therefore EA$ is not perpendicular to BD . Q.E.D. (30)

42. COR. 1. *All right angles are equal.*

For otherwise two could be so placed, as in the figure, that there would be two perpendiculars to the same line at the same point.



43. COR. 2. *All straight angles are equal.* (Ax. 6)

For each is the sum of two right angles. (31)

44. COR. 3. *The complements and supplements of equal angles are equal, and conversely.** (Ax. 3)

45. DEFINITION. If two theorems are so related that the hypothesis and conclusion of the one are respectively the conclusion and hypothesis of the other, the one theorem is said to be the *converse* of the other. Thus, as will presently be seen, Prop. II. and Prop. III. are converse propositions. The converse of a proposition is by no means necessarily true, and, if true, requires to be established by demonstration.

NOTE.—In Prop. I., as in all the propositions of Plane Geometry, it is understood that all the points of each figure lie in one and the same plane. It will afterwards be seen that there may be any number of lines, each perpendicular to a given line at the same point, but not all in the same plane with each other and the line.

EXERCISE 1. If an angle is double its complement, what fraction is it of a right angle? Of a straight angle?

2. If an angle is three times its supplement, what fraction is it of a straight angle? Of a right angle?

3. The lines that bisect adjacent complementary angles form half a right angle.

4. In the diagram for Prop. I., name the complements of the angles CAE and EAD , respectively.

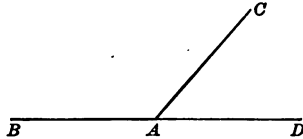
5. In the same diagram, name the supplements of the angles EAD , EAB , and CAB , respectively. The sum of what two of those angles is equal to the supplement of angle CAE ?

6. In the same diagram, suppose EA produced to b . Show that BAb and CAE are complementary angles.

* *Conversely means the converse proposition is also true.* See Art. 45.

PROPOSITION II. THEOREM.

46. If two adjacent angles have their exterior sides in the same straight line, these angles are supplementary.



Given: Two adjacent angles BAC , CAD , with their exterior sides AB , AD , in a straight line;

To Prove: BAC and CAD are supplementary angles.

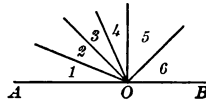
Since AB is in a straight line with AD , (Hyp.)

BAD is a straight \angle . (29)

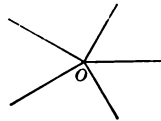
But $\angle BAC + \angle CAD = \angle BAD$, (28)

$\therefore BAC$ and CAD are supplementary \angle s. Q.E.D. (33)

47. COR. 1. The sum of all the angles formed by any number of lines meeting AB , from the same side, in a point O , is equal to a straight angle or two right angles. (Ax: 8)



48. COR. 2. The sum of all the angles formed by any number of lines meeting in a point O is equal to two straight angles or four right angles.



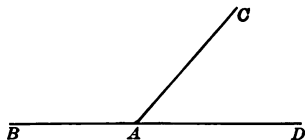
For if any of the lines be produced through O , the sum of the angles on each side of that line will be a st. \angle , (47).

EXERCISE 7. In the diagram for Prop. II., if angle BAC is equal to twice angle CAD , what fraction is each of a right angle?

8. If four lines, OA , OB , OC , OD , meet in a point O , so that angle $AOB =$ angle COD , and angle $AOD =$ angle BOC , then OA and OC are in the same straight line, as are also OB and OD ; and the pairs of angles AOB , AOD , and BOC , DOC , are supplementary.

PROPOSITION III. THEOREM.

49. *If two adjacent angles are supplementary, their exterior sides are in a straight line.*



Given: Two adjacent \angle s, BAC , CAD , that are supplementary;
To Prove: Their exterior sides, AB , AD , are in a straight line.

Since the \angle BAC , CAD , are supplementary, (Hyp.)

$$\angle BAC + \angle CAD = \text{a straight } \angle, \quad (33)$$

\therefore their sum, $\angle BAD$, is a straight \angle ,

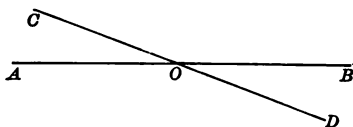
$\therefore AB$ and AD , the sides of $\angle BAD$, are in a st. line. (29)

Q.E.D.



PROPOSITION IV. THEOREM.

50. *If two straight lines intersect, the vertical angles are equal.*



Given: Two straight lines, AB , CD , intersecting in O ;

To Prove: Angle AOC is equal to angle BOD , and angle AOD is equal to angle BOC .

Since $\angle AOD$ is supp. to $\angle AOC$, and also to $\angle BOD$, (46)

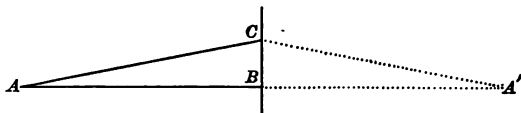
$$\angle AOC = \angle BOD. \quad (44)$$

Similarly, $\angle AOD = \angle BOC$,

each being the supplement of $\angle BOD$. Q.E.D.

PROPOSITION V. THEOREM.

51. *From a point without a line, there can be but one perpendicular to that line.*



Given: The point A , the line BC , $AB \perp$ to BC , and AC any other line from A to BC ;

To Prove: AC is not perpendicular to BC .

Turn the figure ABC about BC till A takes the position A' in the original plane. Mark the point A' , restore ABC to its first position, and join $A'B$, $A'C$.

Then since AB and AC can be made to coincide with $A'B$

and $A'C$, while BC retains its position, (Const.)

$$\angle ABC = \angle A'BC, \text{ and } \angle ACB = \angle A'CB. \quad (25)$$

But $\angle ABC$ is a right \angle , (Hyp.)

$$\therefore \angle A'BC \text{ is a right } \angle, \quad (30)$$

$$\therefore ABA' \text{ is a straight line,} \quad (49)$$

$$\therefore ACA' \text{ is not a straight line,} \quad (6)$$

$$\therefore \angle ACA' \text{ is not a straight } \angle, \quad (29)$$

$$\therefore \angle ACB, \text{ the half of } \angle ACA', \text{ is not a right } \angle. \quad \text{Q.E.D.}$$

EXERCISE 9. If in the diagram for Prop. III., angle CAD is $\frac{1}{2}$ of a right angle, what angle must BA make with AC so that BA shall be in the same straight line with AD ?

10. If, in a line MN , a point P be taken, and two lines PQ , PR , be drawn so that angle $QPM =$ angle RPN , then QR is a straight line.

11. In the diagram for Prop. V., the angles A and A' are the supplements of what angles? Show that these angles are equal.

12. In this diagram, show that the exterior angles at C are equal.

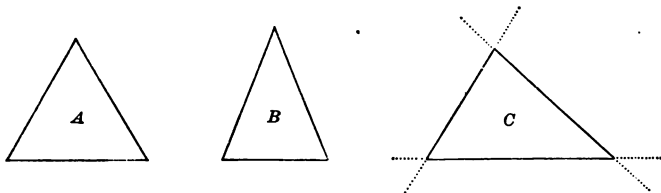
TRIANGLES.

52. A *triangle* is a portion of a plane bounded by three straight lines.

53. The lines that bound a triangle are called its *sides*; the angles formed by the sides, its *angles*; and the vertices of the angles, the *vertices* or *angular points* of the triangle.

If all the sides of a triangle be produced both ways (see triangle *C* below), nine new angles will be formed in addition to those properly called the angles of the triangle, or by way of distinction, its *interior* angles. Of the nine outer angles, the six angles that are supplementary to the interior angles, are called *exterior* angles. Interior angles are always meant whenever we refer to the angles of a triangle without any distinguishing epithet.

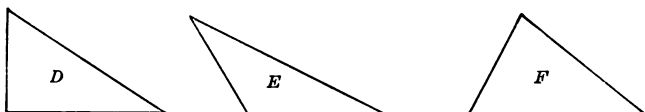
54. The *base* of a triangle is the side on which it is supposed to stand; the angle opposite the base is sometimes referred to as the *vertical* angle.



55. An *equilateral triangle* has three equal sides; as *A*.

56. An *isosceles triangle* has two equal sides; as *B*.

57. A *scalene triangle* has no two sides equal; as *C*.



58. A *right triangle* has a right angle; as *D*.
59. An *obtuse triangle* has an obtuse angle; as *E*.
60. An *acute triangle* has all its angles acute; as *F*.

While lines and angles can be equal only in one way, it is different in regard to triangles and other inclosed figures. For as will afterwards be seen, two triangles that are equal as regards surface may differ greatly as to their sides and angles. Hence it is necessary to define the term *equal* in regard to figures in general.

61. *Equal figures* are such as can be made to coincide exactly; that is, are equal in every respect.

62. **THEOREM.** *Two triangles are equal if their angular points can be made to coincide.*

For if the angular points coincide, the sides terminated by those points must coincide (14); hence, also, the angles formed by the sides, and the surfaces bounded by them, must coincide. It is evident that the theorem may be extended so as to apply to figures bounded by any number of straight lines, the reasoning being exactly the same.

EXERCISE 13. Show, by Prop. V., that no triangle can have two of its angles right angles.

14. If triangle *A* has all its angles equal, show that its six exterior angles are all equal.

15. If a triangle has two equal angles, those angles must be acute. What about the third angle?

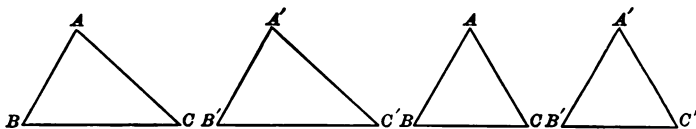
16. If a triangle has no two angles equal, it has three different pairs of equal exterior angles.

17. If, in triangle *D*, the sides containing the right angle be produced through its vertex, the sum of the exterior angles thus formed is equal to a straight angle.

18. If, in triangle *E*, the sides containing the obtuse angle be produced through its vertex, the sum of the exterior angles thus formed is less than a straight angle.

PROPOSITION VI. THEOREM.

63. *Two triangles are equal if a side and the including angles of the one are respectively equal to a side and the including angles of the other.*



Given: In triangles ABC , $A'B'C'$, BC equal to $B'C'$, angle B equal to angle B' , and angle C equal to angle C' ;

To Prove: Triangle ABC is equal to triangle $A'B'C'$.

If $\triangle ABC$ be placed upon $\triangle A'B'C'$, (Post. 5)

so that $BC \neq B'C'$,

then since $\angle B = \angle B'$, (Hyp.)

BA will take the direction of $B'A'$, (25)

and A will lie on $B'A'$ or $B'A'$ produced.

Also since $\angle C = \angle C'$, (Hyp.)

CA will take the direction of $C'A'$, (25)

and A will lie on $C'A'$ or $C'A'$ produced.

Then since A lies on both $B'A'$ and $C'A'$,

A must coincide with A' ,

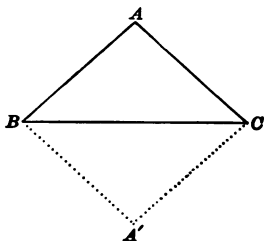
the only point common to $B'A'$ and $C'A'$, (6)

$\therefore \triangle ABC = \triangle A'B'C'$. Q.E.D. (62)

64. **SCHOLIUM.** If the including angles are all equal, i.e., if $\angle C = \angle B = \angle C' = \angle B'$, as in the right hand pair of triangles, it is manifest that $\triangle ABC$ can be made to coincide with $\triangle A'B'C'$ in the reverse position also, so that AB will coincide with $A'C'$, and AC with $A'B'$.

PROPOSITION VII. THEOREM.

65. *If two angles of a triangle are equal, the sides opposite those angles are also equal.*



Given: In triangle ABC , angle C equal to angle B ;

To Prove: AB is equal to AC .

Turn $\triangle ABC$ about its base BC till A takes the position A' (Post. 5). Mark the point A' , restore ABC to its first position, and join $A'B$, $A'C$ (Post. 1), so as to form the $\triangle A'BC$.

Since their angular points can be made to coincide, (Const.)

$$\triangle ABC = \triangle A'BC, \quad (62)$$

and $A'B = AB$, $A'C = AC$, $\angle A'BC = \angle B$, $\angle A'CB = \angle C$. (14)

Since these four angles are all equal, (Hyp. and Const.)

$\triangle ABC$ can be made to coincide with $\triangle A'BC$

in the reverse of the first position, (64)

so that $AB \neq A'C$ and $AC \neq A'B$.

Then since $AB = A'C$, (14)

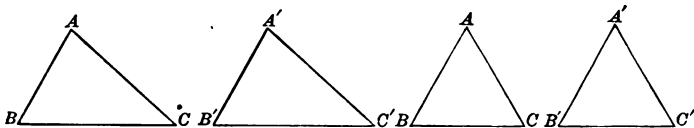
and $AC = A'B$, (Above)

$$AB = AC. \quad \text{Q.E.D.} \quad (\text{Ax. 1})$$

EXERCISE 19. In the diagram for Prop. VII., if AA' be drawn, cutting BC in O , show that when triangle ABC is folded over on triangle $A'BC$, AO must coincide with $A'O$.

PROPOSITION VIII. THEOREM.

66. *Two triangles are equal if two sides and the included angle of the one are respectively equal to two sides and the included angle of the other.*



Given: In triangles ABC , $A'B'C'$, AB equal to $A'B'$, AC equal to $A'C'$, and angle A equal to angle A' ;

To Prove: Triangle ABC is equal to triangle $A'B'C'$.

If $\triangle ABC$ be placed upon $\triangle A'B'C'$, (Post. 5)

so that $\angle A \neq \angle A'$,

then since $AB = A'B'$, (Hyp.)

$B \neq B'$.

Also since $AC = A'C'$, (Hyp.)

$C \neq C'$,

$\therefore \triangle ABC = \triangle A'B'C'$. Q.E.D. (62)

67. **SCHOLIUM.** If the including sides are all equal, *i.e.*, if $AB = AC = A'B' = A'C'$, as in the right hand pair of triangles, it is manifest that $\triangle ABC$ can be made to coincide with $\triangle A'B'C'$ in the reverse position also, so that B will coincide with C' , and C with B' .

EXERCISE 20. Show (Exercise 19) that AO is perpendicular to BC , bisects angle A , and bisects base BC .

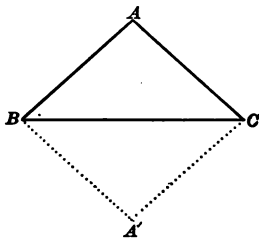
21. Prove, by Prop. VI., that all the triangles having their vertices at O are equal.

22. Prove the same equalities by means of Prop. VIII.

23. In the triangles ABC , $A'B'C'$, above, when AB is made to coincide with $A'B'$, where would AC fall (1) if angle A were greater than angle A' ; (2) if less than angle A' ?

PROPOSITION IX. THEOREM.

68. *If two sides of a triangle are equal, the angles opposite them are equal.*



Given: In triangle ABC , AC equal to AB ;

To Prove: Angle B is equal to angle C .

Turn $\triangle ABC$ about its base BC till A takes the position A' (Post. 5). Mark the point A' , restore ABC to its first position, and draw $A'B$, $A'C$ (Post. 1).

Since their angular points can be made to coincide, (Const.)

$$\triangle ABC = \triangle A'BC, \quad (62)$$

and $A'B = AB$, $A'C = AC$, $\angle A'BC = \angle B$, $\angle A'CB = \angle C$. (14)

Since all these sides are equal, (Hyp. and Const.)

$\triangle ABC$ can be made to coincide with $\triangle A'BC$

in the reverse of the first position, (67)

so that $\angle B \neq \angle A'CB$, and $\angle C \neq \angle A'BC$.

Then since $\angle B = \angle A'CB$, (14)

and $\angle C = \angle A'CB$, (Above)

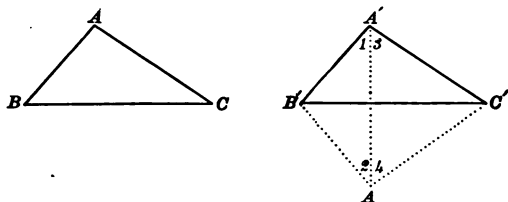
$$\angle B = \angle C. \quad \text{Q.E.D. (Ax. 1)}$$

EXERCISE 24. If two sides of a triangle are unequal, the angles opposite to them are also unequal, and conversely.

25. A scalene triangle cannot have two equal angles.

PROPOSITION X. THEOREM.

69. *Two triangles are equal if the three sides of the one are respectively equal to the three sides of the other.*



Given : In triangles ABC , $A'B'C'$, AB equal to $A'B'$, AC equal to $A'C'$, and BC equal to $B'C'$;

To Prove : Triangle ABC is equal to triangle $A'B'C'$.

Apply $\triangle ABC$ to $\triangle A'B'C'$,

so that $BC \neq B'C'$, and

let A fall on the further side of $B'C'$ from A' .

Join AA' , and first suppose AA' cuts $B'C'$.

Since $AB' = A'B'$, and $AC' = A'C'$, (Hyp.)

$\angle 1 = \angle 2$, and $\angle 3 = \angle 4$, (68)

$\therefore \angle 1 + \angle 3 = \angle 2 + \angle 4$, (Ax. 2)

i.e., $\angle BAC = \angle B'A'C'$,

$\therefore \triangle ABC = \triangle A'B'C'$. Q.E.D. (66)

If AA' should pass through an extremity of $B'C'$, or cut $B'C'$ produced, the proof would be similar to that above. But, as will afterwards be seen, the triangles can always be applied to each other so that AA' shall cut $B'C'$ between B' and C' .

70. SCHOLIUM. In equal triangles, sides that are equal to each other subtend equal angles, and equal angles are subtended by equal sides.

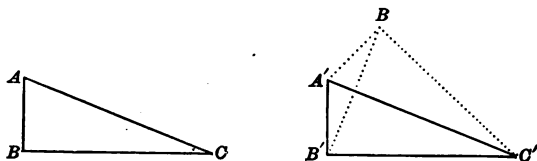
For when equal triangles are made to coincide, coinciding sides necessarily subtend coinciding angles.

This important proposition is a *scholium* rather than a *corollary*, because it does not deduce a new truth, but merely calls attention to a truth, or set of truths, already virtually proved. See definition, p. 19.

71. DEFINITION. In a right triangle, the side subtending the right angle is called the *hypotenuse*; and the other sides are called *arms*.

PROPOSITION XI. THEOREM.

72. *Two right triangles are equal if the hypotenuse and an arm of the one are respectively equal to the hypotenuse and an arm of the other.*



Given: In right triangles ABC , $A'B'C'$, hypotenuse AC equal to hypotenuse $A'C'$, and AB equal to $A'B'$;

To Prove: Right triangle ABC is equal to right triangle $A'B'C'$.

Apply $\triangle ABC$ to $\triangle A'B'C'$, so that $AC \neq A'C'$,
and B falls remote from B' . Join BB' .

Since $AB = A'B'$, (Hyp.)

$\angle A'B'B = \angle A'BB'$, (68)

$\therefore \angle C'B'B = \angle C'BB'$, (44)

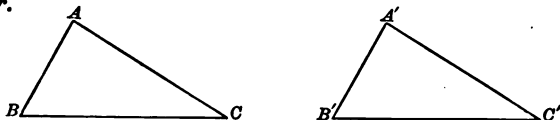
(being complements of equal \angle s,)

$\therefore BC' \text{ or } BC = B'C'$, (65)

$\therefore \triangle ABC = \triangle A'B'C'$. Q.E.D. (69)

PROPOSITION XII. THEOREM.

73. *Two right triangles are equal if the hypotenuse and an acute angle of the one are respectively equal to the hypotenuse and an acute angle of the other.*



Given: In right triangles ABC , $A'B'C'$, hypotenuse BC equal to hypotenuse $B'C'$, and angle B equal to angle B' ;

To Prove: Right triangle ABC is equal to right triangle $A'B'C'$.

Place $\triangle ABC$ upon $\triangle A'B'C'$, so that $BC \neq B'C'$.

Then since $\angle B = \angle B'$, (Hyp.)

BA will take the direction of $B'A'$, (25)

and A will fall upon A' ;

since there can be but one \perp from C' to $B'A'$; (51)

$\therefore \triangle ABC = \triangle A'B'C'$. Q.E.D. (62)

The student is now in possession of all the theorems concerning equal triangles that are of practical importance, viz. —

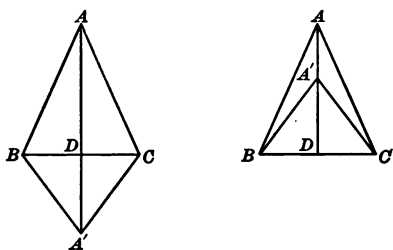
Two triangles are equal if they have, respectively:

1. *One side and the including angles equal;* Prop. VI.
2. *Two sides and the included angle equal;* Prop. VIII.
3. *Three sides mutually equal.* Prop. X.
4. *If right, the hypotenuse and an arm, or the hypotenuse and an acute angle, mutually equal.* Props. XI., XII.

It may be easily proved that triangles are equal that have a side and *any* two angles equal; the case is, however, of no importance, nor is the more difficult one concerning triangles having two sides and a *not* included angle mutually equal.

PROPOSITION XIII. THEOREM.

74. *The line that joins the vertices of two isosceles triangles on the same base, bisects, 1°, the vertical angles, 2°, the common base at right angles.*



Given: AA' joining the vertices of isosceles triangles ABC , $A'BC$, and cutting BC in D ;

To Prove: AA' bisects angles A and A' , and is perpendicular to BC at its mid point.

1°. As $AB = AC$, $A'B = A'C$, and AA' is common,* (Hyp.)

$$\triangle BAA' = \triangle CAA', \quad (69)$$

$\therefore \angle BAD = \angle CAD$, and $\angle BA'D = \angle CA'D$. Q.E.D. (70)

2°. As $AB = AC$, AD is common, and $\angle BAD = \angle CAD$, (1°)

$$\triangle BAD = \triangle CAD, \quad (66)$$

$\therefore BD = CD$, and the \angle s at D are equal, (70)

$\therefore AA'$ is \perp to BC at its mid point. Q.E.D. (30)

75. COR. 1. *If two points are each equidistant from the extremities of a given line, the line joining these points is perpendicular to the given line at its mid point.*

The points, it will be observed, may be on different sides of the given line, or both on the same side.

* That is, is common to (or belongs to) both triangles.

76. DEFINITION. The line that divides an angle into two equal angles is called its *bisector*.

77. COR. 2. *The bisector of the vertical angle of an isosceles triangle bisects the base at right angles; and conversely.*

Thus far the constructions required as aids in the demonstration of theorems have called for nothing beyond the postulates. We shall presently have occasion for other constructions, such as bisection of lines, angles, etc. The problem about to be given is of some importance as entering largely into subsequent problems.

EXERCISE 26. In the diagram for Prop. IX., show that angle $ABA' =$ angle ACA' .

27. Prove, by means of Prop. X., the equality proved in Exercise 22.

28. Draw the diagram for Prop. X. when the angle B is obtuse, and prove the proposition.

29. Prove that on the same side of the same base there can be but one isosceles triangle having its arms, or equal sides, each equal to a given line.

30. Prove that on the same side of the same base there can be but one isosceles triangle having a given vertical angle.

31. In the diagram for Prop. XIII., prove in regard to both figures that the angle ABA' is equal to the angle ACA' .

32. If the base of an isosceles triangle ABC be produced both ways to D and E , so that $BD = CE$, and A be joined with D and E , the triangle ADE will be isosceles.

33. If the exterior angles formed by producing both ways a side of a triangle are equal, the other sides are equal.

34. Right triangles are equal if their arms are respectively equal.

35. Right triangles are equal if they have equal acute angles at the extremities of equal arms.

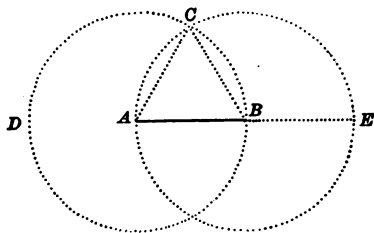
36. In the diagram for Prop. XIV., if BA be produced to meet the circumference at D , show that DE is equal to the sum of the three sides of triangle CAB .

37. In the same diagram, BA being produced as above, and C joined with D and E , show that CAD , CBE , are equal isosceles triangles.

38. In the same diagram, if the circumferences intersect a second time at F , and CF cut AB in G , show that $GE = 3 AG$.

PROPOSITION XIV. PROBLEM.

78. To find a point equidistant from the extremities of a given line.



Given : **A straight line AB ;**

Required : **To find a point equidistant from A and B .**

From A as center, with radius AB , describe the circumference BCD .	} Post. (4)
From B as center, with radius BA , describe the circumference ACE .	

Produce AB to meet ACE in E . (Post. 2)

Since $AE > AB$, (Const.)

E lies without the circumf. BCD . (23)

But A lies within the circumf. BCD , (21)

\therefore circumf. ACE must intersect circumf. BCD .

Let them intersect in C . C is the point required.

For since $AC = AB$,	} (23)
and $BC = BA$ or AB ,	

$\therefore AC = BC$, (Ax. 1)

\therefore a point C has been found equidistant from A and B . Q.E.F.

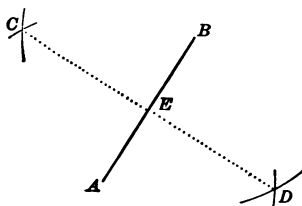
79. SCHOLIUM. Although, for convenience of demonstration, the radii have been taken each equal to the given line, it is manifest that other points equidistant from A and B

can be obtained by taking any equal radii greater than one half the given line. It is also obvious that, in practice, instead of entire circumferences, only such portions need be described as will give the points of intersection.



PROPOSITION XV. PROBLEM.

80. To bisect a given straight line.



Given: A straight line AB ;

Required: To bisect AB ; that is, to find its mid point.

Find two points C and D , each equidistant from A and B . (78)

Join CD . CD will intersect AB at its mid point.

Since C and D are two points equidistant from A and B ,

CD is \perp to AB at its mid point, say E ; (75)

i.e., AB is bisected in E . Q.E.F.

SCHOLIUM. This construction gives not only the mid point of AB , but also the perpendicular to AB through its mid point.

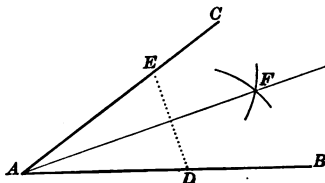
EXERCISE 39. Find a point X that shall be twice as far from each of two given points, A and B , as A is from B .

40. Find a second point Y that shall be three fourths the distance from each of those same points, A and B , that A is from B .

41. Join AB , and prove that the perpendicular at the mid point of AB will pass through X and Y .

PROPOSITION XVI. PROBLEM.

81. To bisect a given angle.



Given: An angle BAC ;

Required: To bisect angle BAC .

On AB , AC , lay off any equal parts AD , AE . (Post. 3)
Join DE (Post. 1), and find a point F equidistant from D and E . (78)

The line joining AF is the bisector of the given angle.

Since A and F are each equidistant from D and E , (Const.)

AF is \perp to DE at its mid point, (75)

$\therefore \angle BAC$ is bisected by AF . Q.E.F. (77)

82. SCHOLIUM. By repeating the operations described in Props. XV. and XVI., a given line or angle can be divided into 4, 8, 16, ..., 2^n , equal parts.

EXERCISE 42. In the diagram above, show that angle $BDE =$ angle CED .

43. Divide a given angle ABC into four equal angles.

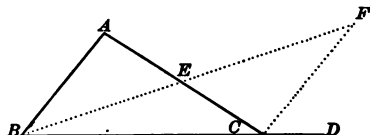
44. In the diagram above, if the point F were taken on the same side of DE as A , what three positions might F have in regard to A ? In which of these positions would the point taken fail to assist in the required construction?

45. In the diagram above, join F with D and E , and prove the proposition by Prop. X.

46. Show that in an equilateral triangle the three lines joining the vertices with the mid points of the opposite sides are (1) equal to each other; (2) perpendicular to these sides; (3) bisectors of the angles.

PROPOSITION XVII. THEOREM.

83. *An exterior angle of any triangle is greater than either of the remote interior angles.*



Given: ACD , an exterior angle of triangle ABC ;

To Prove: Angle ACD is greater than angle A or angle B .

Of the two, let $\angle A$ be not less than $\angle B$.*

Bisect AC in E (80); join BE , and produce BE to F , so that $EF = BE$ (Posts. 1 and 3). Join FC .

Then since $EA = EC$, and $EF = EB$, (Const.)

also $\angle AEB = \angle CEF$, (50)

$\triangle EAB = \triangle ECF$, (66)

$\therefore \angle ECF = \angle A$. (70)

But $\angle ACD > \angle ECF$, (Ax. 8)

$\therefore \angle ACD > \angle A$;

and since $\angle B$ is not greater than $\angle A$, (Hyp.)

$\angle ACD > \angle B$, as well as $> \angle A$. Q.E.D.

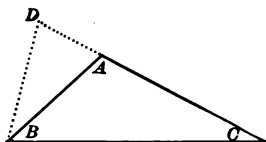
84. COR. 1. *No two angles of a triangle can be supplementary.* For each is less than the exterior angle which is the supplement of the other.

85. COR. 2. *If a triangle has an obtuse or a right angle, each of the others is acute.*

* That is, angle B is, at most, equal to angle A .

PROPOSITION XVIII. THEOREM.

86. *A greater side of a triangle subtends a greater angle.*



Given: In triangle ABC , BC greater than AC ;

To Prove: Angle A is greater than angle B .

Produce CA to D , so that $CD = CB$; and join BD .

Since $CB = CD$, (Const.)

$\angle D = \angle DBC$. (68)

But $\angle A > \angle D$, (83)

(angle A being an exterior \angle of $\triangle DAB$.)

$\therefore \angle A > \angle DBC$,

$\therefore \angle A > \angle B$. Q.E.D. (a.f.)*

87. COR. *Conversely, a greater angle is subtended by a greater side.*

In $\triangle ABC$ suppose $\angle A > \angle B$. AC cannot be equal to BC ; for then $\angle B$ would be equal to $\angle A$ (65): nor can AC be greater than BC ; for then $\angle B$ would be greater than $\angle A$ (86). Hence, since BC can neither be equal to nor less than AC , it must be greater than AC . Q.E.D.

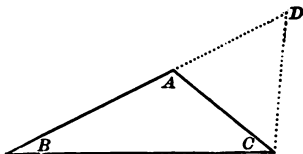
EXERCISE 47. Draw the diagram for Prop. XVII., when the exterior angle ACD is acute.

48. Prove that a perpendicular from a vertex to the opposite side of a triangle falls (1) within the triangle if the angles including that side are both acute; (2) without the triangle if one of those angles is obtuse.

* If $m > n$ and $n > p$, then *a fortiori* (i.e., with greater reason), $m > p$. This is called the argument *a fortiori*, abbreviated a.f.

PROPOSITION XIX. THEOREM.

88. *Any side of a triangle is less than the sum of the other two.*



Given: Any side BC of a triangle ABC ;

To Prove: BC is less than $AB + AC$.

Produce BA to D , so that $AD = AC$; and join DC .

Since $AC = AD$, (Const.)

$\angle ACD = \angle D$. (68)

But $\angle BCD > \angle ACD$, (Ax. 8)

$\therefore \angle BCD > \angle D$,

$\therefore BD$, or $BA + AC$, $> BC$. Q.E.D. (86)

89. COR. *Any side of a triangle is greater than the difference of the other two.*

For since $BC + AC > AB$, (88)

$BC > AB - AC$. (Ax. 5)

EXERCISE 49. Prove Prop. XVIII. by cutting off on BC , CD equal to CA , joining AD , etc.

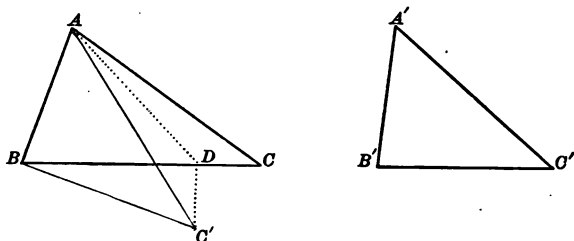
50. The perpendicular to the greatest side from the opposite vertex falls within the triangle.

51. Prove Prop. XIX. by supposing BC to be the greatest side, and drawing AD perpendicular to BC .

52. In a triangle ABC , having AB less than AC , prove that any point in the line joining A with the mid point of BC is nearer to B than to C .

PROPOSITION XX. THEOREM.

90. *If two triangles have two sides of the one respectively equal to two sides of the other, but the included angles unequal, the greater angle is subtended by a greater base.*



Given: Two triangles ABC , $A'B'C'$, having AB equal to $A'B'$, AC equal to $A'C'$, but angle A greater than A' ;

To Prove: Base BC is greater than base $B'C'$.

Place $\triangle A'B'C'$ upon $\triangle ABC$, so that $A'B' \neq AB$. Then
 $\therefore \angle A > \angle A'$ (Hyp.), $A'C'$ will fall between AB and AC , as AC' .

Draw AD bisecting $\angle CAC'$ to meet BC in D , and join DC' .

Since $AC' = AC$ (Hyp.), and AD is common,
 also $\angle DAC' = \angle DAC$, (Const.)

$\therefore \triangle ADC' = \triangle ADC$ (66), and $DC' = DC$. (70)

But $BD + DC' > BC'$, (88)

and $BD + DC = BC$, (Const.)

$\therefore BC > BC'$ or $B'C'$. Q.E.D.

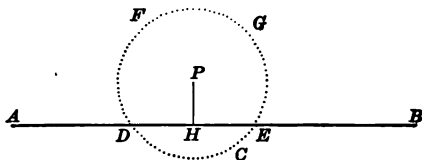
91. COR. Conversely, if triangles ABC , $A'B'C'$, have AB , AC , equal to $A'B'$, $A'C'$, respectively, but the base BC greater than the base $B'C'$, then the angle A is greater than the angle A' .

For $\angle A$ cannot be equal to $\angle A'$, since then BC would be equal to $B'C'$ (66); nor can $\angle A$ be less than $\angle A'$, since then BC would be less than $B'C'$ (90). Hence $\angle A$ must be greater than $\angle A'$.

PERPENDICULARS.

PROPOSITION XXI. PROBLEM.

92. *To draw a perpendicular to a given line from a given point without it.*



Given : A point P without a straight line AB of indefinite length; *

Required : To draw a perpendicular from P to AB .

Take any point C on the side of AB remote from P .

From P as center, with radius PC , describe a circumference, CFG . (Post. 4)

Since AB extends indefinitely between P and C , and the circumference may be described in either direction from C , it will intersect AB in two points, D , E .

Bisect DE in H (80), and join PH . PH is \perp to AB .

Since D and E are points in the circumf., CFG , (Const.)

P is equidistant from D and E ; (21)

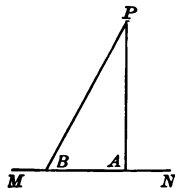
also H is equidistant from D and E , (Const.)

$\therefore PH$ is \perp to DE at its mid point H , (75)

i.e., PH is drawn \perp to AB . Q.E.F.

93. COR. *The perpendicular is the shortest line that can be drawn to a given line from a given point without it.*

For if PA is a perpendicular, and PB a line oblique to MN , then $\angle B < \text{right } \angle A$ (85); hence $PA < PB$ (86).



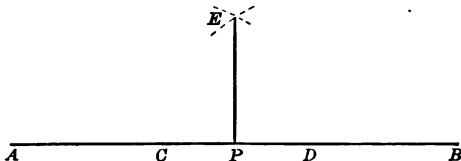
* That is, the given line may be produced to any extent, if necessary.

94. DEFINITION. The *distance of a point from a line* is the distance from the point to the foot of the perpendicular to the line. Thus the distance of P from MN is the perpendicular distance PA , this being the *least* distance from P to any point in MN (93).



PROPOSITION XXII. PROBLEM.

95. *To draw a perpendicular to a given line at a given point in it.*



Given: A point P in a straight line AB ;

Required: To draw a perpendicular to AB at P .

On AB lay off, on each side of P , equal parts PC, PD .

Find a point E that is equidistant from C and D . (78)

Join EP ; then EP is \perp to AB at P .

Since P is equidistant from C and D ,
and E is equidistant from C and D , } (Const.)

EP is \perp to CD at its mid point P , (75)

i.e., EP is drawn \perp to AB at P . Q.E.F.

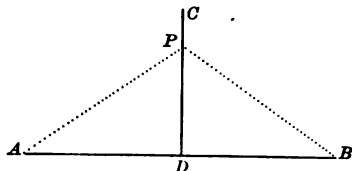
SCHOLIUM. In this problem we have what may be regarded as a special case of the bisection of an angle. What is required is, in fact, to bisect a straight angle (compare 81).

EXERCISE 53. At a given point in a line make an angle equal to half a right angle.

54. From a given point without a line draw a line making with the given line an angle equal to half a right angle.

PROPOSITION XXIII. THEOREM.

96. *Any point in the perpendicular at the mid point of a line is equidistant from the extremities of the line.*



Given: Any point P in CD , the \perp at the mid point of AB ;

To Prove: P is equidistant from A and B .

Join P with A and B .

Then since $DA = DB$ (Hyp.), and PD is common,

$$\text{also rt. } \angle PDA = \text{rt. } \angle PDB, \quad (42)$$

$$\triangle PDA = \triangle PDB, \quad (66)$$

$$\therefore PA = PB, \quad (70)$$

i.e., P is equidistant from A and B . Q.E.D.

97. COR. 1. *Conversely, any point that is equidistant from the extremities of a line will lie in the perpendicular through its mid point.*

For the line joining that point with the mid point of the line is perpendicular to the latter at its mid point (75).

98. COR. 2. *Any point not in the perpendicular through the mid point of a line is unequally distant from its extremities.*

For if it were equidistant from those extremities, it would be in that perpendicular (97).

EXERCISE 55. In the diagram for Prop. XXIII., show that angle $CPA =$ angle CPB .

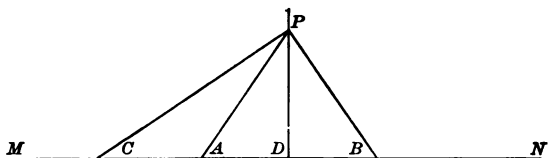
56. In the same diagram, what condition must be fulfilled in order that triangle PAB may be equiangular?

PROPOSITION XXIV. THEOREM.

99. *If from any point in a perpendicular to a given line different oblique lines be drawn to the given line, then*

1°. *Oblique lines drawn to equal distances from the foot of the perpendicular are equal.*

2°. *Of oblique lines drawn to unequal distances from the foot of the perpendicular, the more remote is the greater.*



Given : $PD \perp$ to MN , and PA, PB, PC , drawn oblique to MN , so that AD is equal to BD , but CD is greater than AD ;

To Prove : PA is equal to PB , but PC is greater than PA or PB .

1°. Since P is in the \perp at the mid point of AB , (Hyp.)

$$PA = PB. \quad \text{Q.E.D.} \quad (96)$$

2°. Since exterior $\angle PAC >$ right $\angle D$, (83)

$$\text{but right } \angle D > \angle C, \quad (85)$$

$$\angle PAC > \angle C, \quad (\text{a.f.})$$

$$\therefore PC > PA \text{ or its equal } PB. \quad \text{Q.E.D.} \quad (86)$$

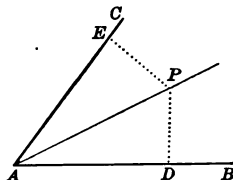
100. COR. *Only two equal straight lines can be drawn from a point to a given straight line, one on each side of the perpendicular from that point.*

EXERCISE 57. From a point without a line, show that only two oblique lines can be drawn so as to make equal angles with the given line.

58. If oblique lines drawn from a given point without it make unequal interior angles with a line, that which makes the lesser angle is the greater.

PROPOSITION XXV. THEOREM.

101. *Any point in the bisector of an angle is equidistant from its sides; and conversely.*



1°. *Given*: A point P in the bisector of angle BAC ;

To Prove: P is equidistant from AB and AC .

Draw $PD \perp$ to AB and $PE \perp$ to AC . Then

since AP is the hypotenuse of rt. $\triangle PDA$, PEA , (Const.)

and $\angle PAD = \angle PAE$, (Hyp.)

right $\triangle PAD =$ right $\triangle PAE$, (73)

$\therefore PD = PE$. Q.E.D. (70)

2°. *Given*: In angle BAC , a point P equidistant from AB and AC ;

To Prove: The line joining PA bisects angle BAC .

Draw $PD \perp$ to AB , $PE \perp$ to AC , and join PA .

Then since $PD = PE$ (Hyp.), and AP is common,

also right $\angle D =$ right $\angle E$, (Const.)

right $\triangle PAD =$ right $\triangle PAE$, (72)

$\therefore \angle PAD = \angle PAE$, (70)

$\therefore PA$ bisects $\angle BAC$. Q.E.D.

EXERCISE 59. Show that Prop. XXIII. is a particular case of XXV.

60. Prove that any point not in the bisector of an angle is unequally distant from its sides.

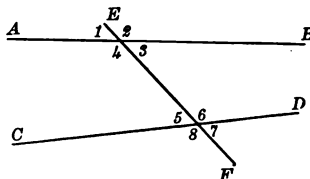
61. Enunciate and prove the converse of Exercise 60.

62. Prove that a perpendicular to the bisector of an angle makes equal angles with the sides.

PARALLELS.

102. *Parallel* lines are such as lie in the same plane, but cannot meet, however far produced in either direction.

103. A *transversal* is a straight line that is transverse to, that is, meets or intersects, a set of two or more straight lines.

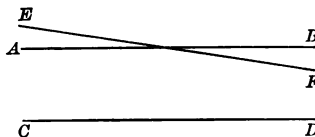


104. When a transversal EF intersects two lines AB , CD , then

- (a) The four \angle s 3, 4, 5, 6, within AB , CD , are *interior \angle s*.
- (b) The four \angle s 1, 2, 7, 8, without AB and CD are *exterior \angle s*.
- (c) The pairs of interior \angle s, 3-5, 4-6, situated on opposite sides of the transversal, are *alternate-interior \angle s*.
- (d) The pairs of exterior \angle s, 1-7, 2-8, situated on opposite sides of the transversal, are *alternate-exterior \angle s*.
- (e) The pairs of \angle s, 1-5, 4-8, 2-6, 3-7, situated on the same side of the transversal, but one exterior, the other interior, are *corresponding \angle s*.

Angles hereafter referred to simply as alternate are to be understood as being alternate-interior, those most frequently mentioned.

105. AXIOM 10. *Two intersecting lines cannot both be parallel to a third line.*



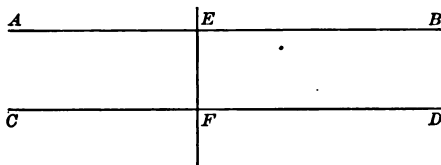
Thus, if AB is parallel to CD , EF cannot be so; and *vice versa*.

EXERCISE 63. Enunciate and prove the converse of Exercise 62.

64. Prove that the bisector of an angle of a triangle divides the opposite side equally or unequally according as the other sides are or are not equal.

PROPOSITION XXVI. THEOREM.

106. *If a transversal is perpendicular to two lines in the same plane, these lines are parallel.*



Given: EF perpendicular to AB and also to CD ;

To Prove: AB is parallel to CD .

Since AB and CD are each \perp to EF , (Hyp.)

AB cannot meet CD in any point on either side of EF , no matter how far produced,

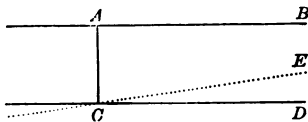
since from the same point there cannot be two \perp s to a line, (51)

$\therefore AB$ is \parallel to CD . Q.E.D. (102)



PROPOSITION XXVII. THEOREM.

107. *If a transversal is perpendicular to one of two parallels, it is perpendicular to the other also.*



Given: Two parallels AB , CD , and AC a transversal \perp to AB ;

To Prove: AC is perpendicular to CD .

At C suppose CE drawn \perp to AC . (95)

Then since CE is \parallel to AB , (106)

and CD is \parallel to AB , (Hyp.)

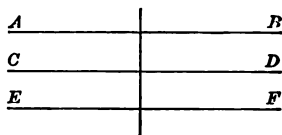
CD must coincide with CE ,

since otherwise there would be through C two intersecting parallels to the same line, which is impossible. (Ax. 10)

$\therefore CD$ is \perp to AC . Q.E.D.

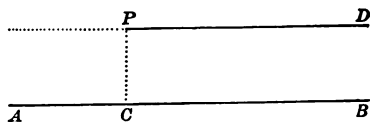
108. COR. *If AB, CD , are each parallel to EF , then AB is parallel to CD .*

For a \perp to EF must be \perp to both AB and CD (107).



PROPOSITION XXVIII. PROBLEM.

109. *Through a given point without a line to draw a parallel to the line.*



Given: A point P without a given line AB ;

Required: To draw through P a line parallel to AB .

From P draw $PC \perp$ to AB , or AB produced, (92)

and from P draw $PD \perp$ to PC . (95)

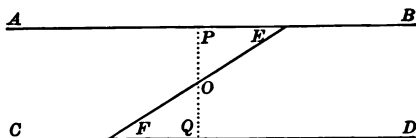
PD is the parallel required.

For since PC is \perp to AB , and also to PD , (Const.)

$\therefore PD$ is \parallel to AB . Q.E.F. (106)

PROPOSITION XXIX. THEOREM.

110. *Two lines are parallel if a transversal to those lines makes the alternate angles equal; and conversely.*



1°. *Given: EF transversal to AB, CD, making the alternate angles E, F, equal;*

To Prove: AB is parallel to CD.

Find O, the mid point of EF (80), draw $OP \perp$ to AB (92), and produce PO to meet CD in Q.

Then since $\angle F = \angle E$ (Hyp.), and $\angle FOQ = \angle EOP$, (50)

and $OF = OE$, (Const.)

$\triangle OQF = \triangle OPE$, (63)

$\therefore \angle Q = \text{rt. } \angle P$, (70)

$\therefore AB$ is \parallel to CD . Q.E.D. (106)

2°. *Given: A transversal EF meeting the parallels AB, CD, in E, F, respectively;*

To Prove: Angle E is equal to alternate angle F.

Making the same constructions as in 1°,

Since AB is \parallel to CD (Hyp.), and PQ is \perp to AB , (Const.)

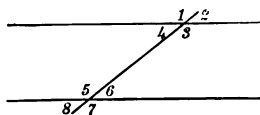
PQ is also \perp to CD (107), and Q is a rt. \angle ;

also $\angle EOP = \angle FOQ$ (50), and $OE = OF$, (Const.)

$\therefore \text{rt. } \triangle OPE = \text{rt. } \triangle OQF$, (73)

$\therefore \angle E = \angle F$. Q.E.D. (70)

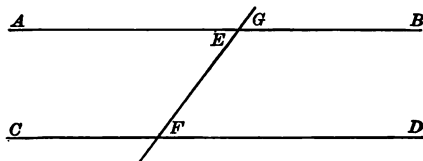
111. SCHOLIUM. It is sufficient to prove the foregoing propositions, as above, for one pair of alternate angles only; since the equality of any one pair of the alternate angles



formed by a transversal and two lines, entails the equality of every pair. For if $\angle 4 = \angle 6$, then $\angle 2 = \angle 8$ (50); also $\angle 3 = \angle 5$, and $\angle 1 = \angle 7$ (44).

PROPOSITION XXX. THEOREM.

112. *Two lines are parallel if a transversal to those lines makes the corresponding angles equal; and conversely.*



1°. *Given*: EF transverse to AB , CD , making the corresponding angles F , G , equal;

To Prove: AB is parallel to CD .

Since $\angle F = \angle G$ (Hyp.), and $\angle E = \angle G$, (50)

$\angle E = \angle F$, (Ax. 1)

$\therefore AB$ is \parallel to CD . Q.E.D. (110')*

2°. *Given*: A transversal EF meeting the parallels AB , CD , in E , F , respectively;

To Prove: Angle F is equal to the corresponding angle G .

Since EF is transverse to the \parallel 's AB , CD , (Hyp.)

$\angle E = \angle F$. (110'')

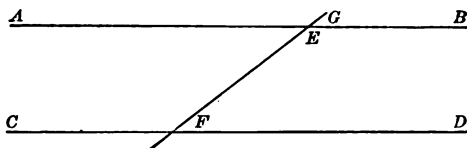
But $\angle E = \angle G$, (50)

$\therefore \angle F = \angle G$. Q.E.D. (Ax. 1)

* When, as here, reference is made to a proposition consisting of two parts, these parts will be distinguished by accents; thus, 110' and 110''.

PROPOSITION XXXI. THEOREM.

113. *Two lines are parallel if a transversal to those lines makes the interior angles on the same side supplementary; and conversely.*



1°. *Given:* EF transverse to AB , CD , making the interior angles E , F , supplementary;

To Prove: AB is parallel to CD .

Since $\angle F$ is the supplement of $\angle E$, (Hyp.)

and $\angle G$ is the supplement of $\angle E$, (46)

$\angle F =$ the corresponding $\angle G$, (Ax. 1)

$\therefore AB$ is \parallel to CD . Q.E.D. (112')

2°. *Given:* A transversal EF meeting the parallels AB , CD , in E , F , respectively;

To Prove: The interior angles E and F are supplementary.

Since EF is transverse to the \parallel 's AB , CD , (Hyp.)

$\angle F =$ corresponding $\angle G$. (112'')

But $\angle E$ is the supplement of $\angle G$, (46)

$\therefore \angle E$ is the supplement of $\angle F$. Q.E.D. (44)

114. COR. *If a transversal to two lines makes the sum of the interior angles on one side less than a straight angle, these lines will meet if produced on that side.*

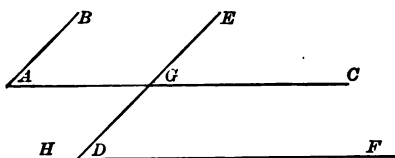
For if they did not meet, they would be parallel, and those interior angles would be supplementary (113''); that they must meet on that side of the transversal on which the interior \angle s are less than a straight \angle follows from Art. 84.

115. DEFINITION. Parallel lines are said to lie in the *same* or *opposite* directions, according as they are on the same or opposite sides of the transversal through the points from which they are supposed to be drawn.



PROPOSITION XXXII. THEOREM.

116. Angles whose corresponding sides are parallel are equal or supplementary.



Given: AB parallel to DE , and AC parallel to HF , forming angles at A and D ;

To Prove: Angle A is equal to angle D , and angle A is supplement of angle EDH .

Let AC , DE , produced if necessary, meet in G .

Since AB is \parallel to DE , and AC is transverse to both,

$$\angle G = \angle A; \quad (112'')$$

Since AC is \parallel to DF , and DE is transverse to both,

$$\angle G = \angle D, \quad (112'')$$

$$\therefore \angle A = \angle D; \quad \text{Q.E.D. (Ax. 1)}$$

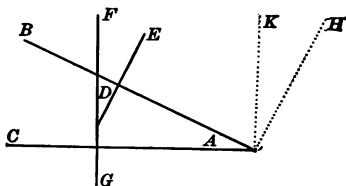
$$\text{also } \angle D \text{ is supp. of } \angle EDH, \quad (46)$$

$$\therefore \angle A \text{ is supp. of } \angle EDH. \quad \text{Q.E.D. (44)}$$

117. SCHOLIUM. The angles are *equal* if both corresponding pairs of sides lie in the *same* or *opposite* directions; they are *supplementary* if one pair has the *same* and the other *opposite* directions.

PROPOSITION XXXIII. THEOREM.

118. *Angles whose corresponding sides are perpendicular to each other are either equal or supplementary.*



Given: AB perpendicular to DE and AC perpendicular to DF , forming angles at A and D ;

To Prove: Angle A is equal to angle D , and angle A is the supplement of angle EDG .

At A draw $AH \perp$ to AB , and $AK \perp$ to AC . (95)

Since BAH and CAK are rt. \angle s, (Const.)

$$\angle A = \angle HAK, \quad (44)$$

(each being the comp. of $\angle BAK$.)

Since AB is \perp to DE and AH , and AC is \perp to DF and AK ,
(Hyp. and Const.)

DE is \parallel to AH , and DF is \parallel to AK ; (106)

and they lie in the same directions,

$$\therefore \angle D = \angle HAK, \quad (117)$$

$$\therefore \angle A = \angle D; \quad \text{Q.E.D. (Ax. 1)}$$

$$\text{also } \angle D \text{ is supp. of } \angle EDG, \quad (46)$$

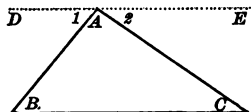
$$\therefore \angle A \text{ is supp. of } \angle EDG. \quad \text{Q.E.D. (44)}$$

119. SCHOLIUM. The angles are equal if both are acute or both obtuse.

EXERCISE 65. In the diagram for Prop. XXIX., show that PF being joined, a line drawn through Q parallel to PF will pass through E .

PROPOSITION XXXIV. THEOREM.

120. *The sum of the angles of any triangle is equal to a straight angle.*



Given: Any triangle ABC ;

To Prove: $\angle A + \angle B + \angle C$ is equal to a straight angle.

Through A draw $DAE \parallel$ to BC . (109)

$\therefore DE$ is \parallel to BC , and AB is transverse to both, (Const.)

$$\angle B = \angle 1; \quad (110'')$$

$\therefore DE$ is \parallel to BC , and AC is transverse to both,

$$\angle C = \angle 2, \quad (110'')$$

$$\therefore \angle A + \angle B + \angle C = \angle A + \angle 1 + \angle 2, \quad (\text{Ax. } 2)$$

$$\text{i.e., } \angle A + \angle B + \angle C = \angle DAE, \text{ a st. } \angle. \quad \text{Q.E.D.} \quad (47)$$

121. COR. 1. *Each angle of a triangle is the supplement of the sum of the other two.*

122. COR. 2. *An exterior angle of a triangle is equal to the sum of the two remote interior angles.*

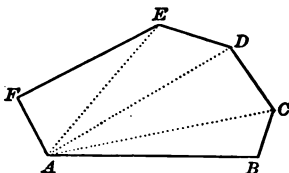
For its supplement is also the supplement of those two.

123. COR. 3. *The acute angles of a right triangle are complementary.*

124. DEFINITION. A *plane polygon* is a portion of a plane bounded by straight lines. Its *sides*, *angles*, and *vertices* are defined as for the triangle, which is a three-sided polygon. The polygons treated of are to be understood as being *convex*; i.e., such that no side, if produced, will fall within the polygon.

PROPOSITION XXXV. THEOREM.

125. *The sum of the interior angles of a polygon is equal to as many straight angles as the figure has sides, less two.*



Given: A polygon $ABCDEF\dots$ of n sides;

To Prove: $\angle A + \angle B + \angle C + \dots = (n - 2)$ straight angles.

From any vertex, as A , draw lines to C, D, E, \dots .

Since, except the two that meet at A , each of the n sides of the polygon is the base of a triangle having its vertex at A , (Const.)

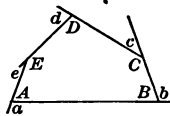
the number of triangles $= n - 2$;

\therefore the sum of the int. \angle s of the $\Delta = (n - 2)$ st. \angle s, (120)

\therefore the sum of the int. \angle s of the polygon $= (n - 2)$ st. \angle s. Q.E.D.

126. COR. 1. *If the sides of a polygon be produced, going round in the same order, the sum of the exterior \angle s thus formed $=$ two straight \angle s.*

For each interior $\angle A$ with its exterior $\angle a$ is equal to a straight \angle (46). Hence the sum of all the angles, interior and exterior, $= n$ straight \angle s. But the interior \angle s alone $= (n - 2)$ straight \angle s; hence the exterior \angle s $=$ two straight \angle s.



127. COR. 2. *Each interior angle of an equiangular polygon is equal to $\frac{n-2}{n}$ straight angles.*

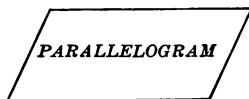
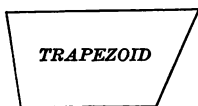
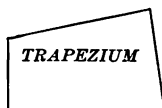
QUADRILATERALS.

128. A *quadrilateral* is a polygon bounded by four sides.

129. A *trapezium* is a quadrilateral having no two of its sides parallel.

130. A *trapezoid* is a quadrilateral having two of its sides parallel.

131. A *parallelogram* is a quadrilateral having its opposite sides parallel.

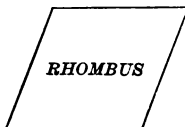
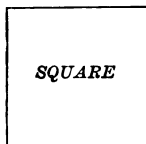
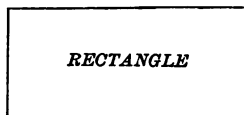


132. A *rectangle* is a right-angled parallelogram.

133. A *square* is an equilateral rectangle.

134. A *rhombus* is a parallelogram which is equilateral but not equiangular.

135. A *diagonal* is a line joining any two nonadjacent vertices of any polygon.



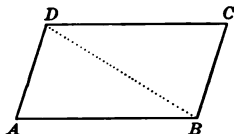
EXERCISE 66. Prove that each angle of an equilateral triangle is one third of a straight angle.

67. Prove that if an angle at the base of an isosceles triangle is one half of the vertical angle, the latter is a right angle.

68. If an angle at the base of an isosceles triangle is n times the vertical angle, what fraction is the latter of a straight angle?

PROPOSITION XXXVI. THEOREM.

136. *The opposite sides and angles of a parallelogram are equal.*



Given: A parallelogram $ABCD$, or AC ;

To Prove: $AB = CD$, $AD = BC$; $\angle A = \angle C$, and $\angle B = \angle D$.

Since AB is \parallel to CD , and BC is \parallel to AD , (Hyp.)
and the corresponding lines lie in opposite directions, (115)

$$\angle A = \angle C, \text{ and } \angle B = \angle D. \quad \text{Q.E.D.} \quad (117)$$

Draw the diagonal DB .

Then since BD is transverse to the \parallel 's AB , CD ,

$$\angle BDC = \angle ABD; \quad (110'')$$

similarly, $\angle CBD = \angle ADB$, and BD is common,

$$\therefore \triangle ABD = \triangle CBD, \quad (63)$$

$$\therefore AB = CD, \text{ and } AD = BC. \quad \text{Q.E.D.} \quad (70)$$

137. DEFINITION. An *intercept* is the straight line, or part of a line, intercepted between two other lines. Thus a diagonal is an intercept through the angular points.

138. COR. 1. *Parallel intercepts between parallels are equal.*

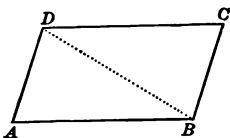
139. COR. 2. *Two parallels are everywhere equally distant.*

140. COR. 3. *A diagonal divides a parallelogram into two equal triangles.*

This was proved in the demonstration of Prop. XXXVI.

PROPOSITION XXXVII. THEOREM.

141. *If the opposite sides or angles of a quadrilateral are equal, the figure is a parallelogram.*



Given : In the quadrilateral $ABCD$,

1°, AB equal to CD , BC equal to AD ;

2°, angle A equal to angle C , angle B to angle D ;

To Prove : In either case, $ABCD$ is a parallelogram.

Draw the diagonal BD . Then

1°. Since $AB = CD$, $AD = BC$, and BD is common, (Hyp.)

$$\triangle ABD = \triangle CBD, \quad (69)$$

$$\therefore \angle ABD = \angle BDC, \text{ and } \angle ADB = \angle CBD, \quad (70)$$

$$\therefore AB \text{ is } \parallel \text{ to } CD, \text{ and } AD \text{ is } \parallel \text{ to } BC, \quad (110')$$

$$\therefore ABCD \text{ is a parallelogram. Q.E.D. } (131)$$

2°. Since $\angle A = \angle C$, and $\angle B = \angle D$, (Hyp.)

$$\angle A + \angle B = \angle C + \angle D. \quad (\text{Ax. 2})$$

$$\text{But } \angle A + \angle B + \angle C + \angle D = 4 \text{ rt. } \angle, \quad (125)$$

$$\therefore \angle A + \angle B = 2 \text{ rt. } \angle, \quad (\text{Ax. 7})$$

$$\therefore AD \text{ is } \parallel \text{ to } BC; \quad (113')$$

similarly, AB is \parallel to CD ,

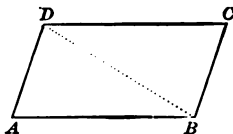
$$\therefore ABCD \text{ is a parallelogram. Q.E.D. } (131)$$

EXERCISE 69. Show that an angle of a triangle is obtuse or acute, according as it is greater or less than the sum of the other angles.

70. Show that the bisectors of opposite angles of a parallelogram are parallel.

PROPOSITION XXXVIII. THEOREM.

142. *If two sides of a quadrilateral are parallel and equal, the figure is a parallelogram.*



Given: In the quadrilateral $ABCD$, AB equal and parallel to CD ;

To Prove: $ABCD$ is a parallelogram.

Draw the diagonal BD .

Then since BD is transverse to the parallels AB , CD ,

$$\angle ABD = \text{alt. } \angle CDB; \quad (110'')$$

also $AB = CD$, and BD is common, (Hyp.)

$$\therefore \triangle ABD = \triangle CBD, \quad (66)$$

$$\therefore \angle ADB = \angle CBD, \quad (70)$$

$$\therefore AD \text{ is } \parallel \text{ to } BC, \quad (110')$$

$$\therefore ABCD \text{ is a parallelogram. } \text{Q.E.D.} \quad (131)$$

EXERCISE 71. A quadrilateral whose diagonals bisect each other is a parallelogram.

72. In a quadrilateral $ABCD$, if angle A is supplementary to angle B , and angle B to angle C , then the figure is a parallelogram.

73. If two parallelograms have an angle of the one equal to an angle of the other, they are mutually equiangular.

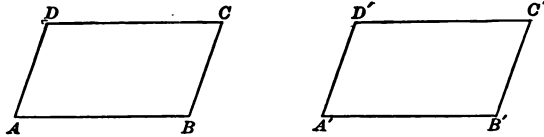
74. A parallelogram whose diagonals are equal is a rectangle.

75. A parallelogram whose diagonals bisect the angles through whose vertices they pass is equilateral.

76. If one angle of a parallelogram is a right angle, all its angles are right angles.

PROPOSITION XXXIX. THEOREM.

143. *Two parallelograms are equal if two sides and the included angle of the one are respectively equal to two sides and the included angle of the other.*



Given: In parallelograms AC and $A'C'$, AB equal to $A'B'$, AD equal to $A'D'$, and angle A equal to angle A' ;

To Prove: Parallelogram AC is equal to parallelogram $A'C'$.

Place AC upon $A'C'$, so that $\angle A \neq \angle A'$.

Since $AB = A'B'$, and $AD = A'D'$, (Hyp.)

B will fall on B' , and D on D' . (14)

$\therefore DC, D'C'$ are \parallel to the coinciding lines $AB, A'B'$, (Hyp.)

DC must take the direction of $D'C'$; }
similarly, BC must take the direction of $B'C'$, } (Ax. 10)

$\therefore C$, the common point of DC, BC , must fall on C' , the common point of $D'C', B'C'$,

\therefore parallelogram $AC =$ parallelogram $A'C'$. Q.E.D. (61)

144. SCHOLIUM. Hence two adjacent sides and the included angle are said to *determine* a parallelogram.

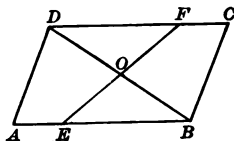
EXERCISE 77. If both diagonals of a parallelogram are drawn, of the four triangles thus formed, those opposite are equal.

78. The bisectors of those angles of a parallelogram that have a side common, meet at right angles.

79. In the diagram for Prop. IV., if OA be made equal to OB , and OC to OD , and AC, AD, BC, BD , be drawn, $ABCD$ will be a parallelogram. What condition must be fulfilled so that the figure will be equilateral?

PROPOSITION XL. THEOREM.

145. *An intercept passing through the mid point of a diagonal of a parallelogram is bisected in that point, and cuts off equal parts on the intercepting sides.*



Given: In parallelogram AC , an intercept EF through O , the mid point of diagonal BD ;

To Prove: OE is equal to OF , and EB to FD .

Since $OB = OD$, (Hyp.)

also $\angle BDF = \angle DBE$ ($110''$), and \angle at O are equal, (50)

$\triangle BOE = \triangle DOF$, (63)

$\therefore OE = OF$, and $EB = FD$. Q.E.D. (70)

146. COR. *The diagonals of a parallelogram bisect each other.*

EXERCISE 80. In the diagram for Prop. XV., if A and B be each joined with C and D , then $ADBC$ will be a square or a rhombus according as EA is or is not equal to EC .

81. If two equal isosceles triangles are constructed on opposite sides of the same base, what sort of a figure is obtained?

82. Each interior angle of an equiangular polygon of six sides is the double of each interior angle of an equilateral triangle.

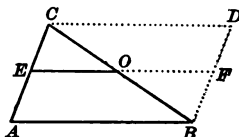
83. How many sides has an equiangular polygon if an interior is equal to one half of an exterior angle?

84. How many sides has an equiangular polygon if an interior is the double of an exterior angle?

85. In the diagram for Prop. XL., show that $OEAD = OFCB$.

PROPOSITION XLI. THEOREM.

147. *An intercept parallel to the base of a triangle and bisecting one side bisects the other also.*



Given: In triangle ABC , an intercept OE parallel to AB and bisecting BC in O ;

To Prove: AC is bisected in E .

Complete the parallelogram AD by drawing BD , CD , parallel to AC , AB , respectively, and produce EO to meet BD in F .

Since EF is an intercept through the mid point of BC , (Hyp.)

$$BF = EC. \quad (145)$$

$$\text{But } BF = AE, \quad (136)$$

(since AF is a parallelogram by construction,)

$$\therefore AE = EC. \quad \text{Q.E.D. (Ax. 1)}$$

148. COR. *Conversely, if OE bisects both AC and BC , then OE is parallel to AB , and OE is equal to $\frac{1}{2} AB$.*

For OE must coincide with the parallel to AB through O , since that parallel must pass through E , the mid point of BC (147). Also OE is equal to $\frac{1}{2} FE$ (145), and FE is equal to AB (136).

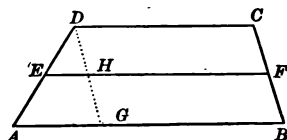
EXERCISE 86. In the diagram for Prop. XLI., show that $OEAB$ can be superposed on $OFDC$.

87. In the diagram for Prop. XLII., if BD , CG , be drawn, show that these lines will intersect in the mid point of FH .

88. In the same diagram, if $AD = BC$, then the angles A and B are equal.

PROPOSITION XLII. THEOREM.

149. *An intercept parallel to the bases of a trapezoid and bisecting one of the nonparallel sides bisects the other also.*



Given: In trapezoid AC , EF parallel to AB , bisecting AD in E and meeting BC in F ;

To Prove: BC is bisected in F .

Draw $DG \parallel$ to BC , meeting AB , EF , in G , H , resp.

Since in $\triangle DAG$, EH is \parallel to AG and bisects AD , (Hyp.)

DG is also bisected in H . (147)

But BH , HC , are parallelograms, (Const.)

$\therefore BF = GH$, and $FC = HD$ or GH , (136)

$\therefore BF = FC$. Q.E.D. (Ax. 1)

150. COR. *Conversely, if EF bisects both AD and BC , then EF is parallel to AB and equal to $\frac{1}{2}(AB + CD)$.*

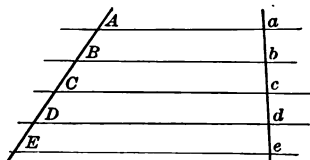
For EF must coincide with the parallel to AB through E , since that parallel must pass through F , the mid point of BC (149). Also $EF = EH + HF = \frac{1}{2}AG + \frac{1}{2}(BG + CD) = \frac{1}{2}(AB + CD)$.

EXERCISE 89. In the diagram for Prop. XLII., if CK be drawn parallel to AD , then will BK be equal to AG , and $GK = 2DC - AB$.

90. If through P , the mid point of AE , PQ be drawn parallel to AB to meet BF in Q , then will BQ be one fourth of BC , and PQ will be equal to $\frac{1}{4}(3AB + CD)$.

PROPOSITION XLIII. THEOREM.

151. *If a series of parallels make equal intercepts on one transversal, they make equal intercepts on any transversal.*



Given: Transversals AE , ae , cut by a series of parallels Aa , Bb , Cc , Dd , Ee , so that the intercepts AB , BC , CD , DE , on AE are equal;

To Prove: The intercepts ab , bc , cd , de , on ae , are also equal.

If AE is \parallel to ae , the opposite intercepts are equal. (136)

If AE is not \parallel to ae , then in the trapezoid $ACca$,
since Bb is \parallel to the bases Aa , Cc , and bisects AC , (Hyp.)

$$ab = bc. \quad (149)$$

In the same way may be proved that $bc = cd$, and $cd = de$.

$$\therefore ab = bc = cd = de. \quad \text{Q.E.D. (Ax. 1)}$$

152. COR. *Conversely, if Aa , Bb , etc., make equal intercepts on AE and also on ae , then Aa is \parallel to Bb , Cc , etc.* (150)

EXERCISES.

QUESTIONS.

91. Would a triangle constructed of rods hinged at their extremities be rigid; that is, incapable of change of form? Would a parallelogram similarly constructed be rigid?

92. Triangles having their sides severally equal have their angles also severally equal. Is the converse true?

93. An acute angle being given, by what construction can you find its complement? Its supplement?



94. By what angle is the supplement of an angle greater than its complement?

95. A right angle being 90° , how many degrees are there in the complement of an angle of 36° ? Of 45° ? Of 90° ? How many in their supplements?

96. Bisect an obtuse angle (81); also a straight angle. Can you bisect a reflex angle, *i.e.*, an angle greater than a straight angle, by the same process?



97. In Proposition XXIII., can you prove Cor. 2 independently?

98. Are all straight lines that cannot meet parallel?

99. If one angle of a triangle is 23° , what is the sum of the other angles?

100. If one angle of a triangle is equal to the sum of the other two, how many degrees has that angle? What is it called?

101. If the vertical angle of an isosceles triangle is 50° , how many degrees in each of the base angles?

102. If a base angle of an isosceles triangle is 45° , what is the vertical angle?

103. In an isosceles triangle, if each base angle is (1) twice, (2) three times, (3) n times, the vertical angle, how many degrees in each?

104. How many degrees in each angle of an equilateral triangle?

105. An acute angle of a right triangle is $\frac{1}{5}$ of the other acute angle. How many degrees in each? Generalize by putting $\frac{m}{n}$ for $\frac{1}{5}$.

106. Arrange the following terms in order of generality: *square*, *polygon*, *rectangle*, *quadrilateral*, *parallelogram*.

107. I wish to cut off half a rectangular field by a straight fence. Through what point must the fence pass?

108. The base of a triangle is fifty feet. How long is the line joining the mid points of the two sides?

109. How many degrees in the sum of the interior angles of a polygon of four sides? Of five? Of six? Of n sides?

110. How many degrees in each interior angle of an equiangular polygon of four sides? Of five? Of six? Of ten sides?

111. One angle of a parallelogram is double the other. How many degrees in each? How many, if one is $\frac{m}{n}$ of the other?

GEOMETRICAL SYNTHESIS AND ANALYSIS.

In the demonstration of propositions we have usually proceeded by the method of *synthesis*, or *direct proof*; that is, taking as a basis certain admitted truths, we built upon these a demonstration of the truth we wished to establish. In other words, in synthesis we reason from admitted principles to consequences (see, for example, 72). This, however, though usually the most convenient way of presenting the proof of a proposition, does not show *how* that proof was invented; we are given a result, not the process by which the result was reached.

In the method of *analysis* we proceed in the opposite way; that is, assuming as true the conclusion we wish to establish, we reason back to principles. If we are led back to principles already known as true, we can take these as the basis of a synthetic proof of the conclusion we wish to establish. If, on the other hand, we are led to a contradiction of a known truth, we know that the assumed conclusion was false.

This indirect method is often made use of in the demonstration of theorems for which it would be inconvenient or difficult to find direct proof. In Prop. XXVII., for example, we show that CD must coincide with CE because their noncoincidence would entail a consequence that had been shown to be impossible. Instead of proving *directly* the conclusion we wish to establish, we prove it *indirectly* by showing that any other conclusion would lead to a contradiction of some known truth.

Among the exercises about to be given, most are so easy that the principles on which to base a synthetic proof at once suggest themselves. The student should, however, in every case, base his synthetic proof upon a previous analysis, guided by the general directions given below. The diagrams and references given as aids should, as far as possible, be left as a last resort.

ANALYSIS OF THEOREMS.

1. Assuming the theorem as true, construct a diagram accordingly.

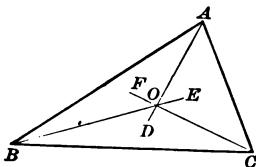
2. Deduce, with the aid of such constructions as may be necessary, any consequences that follow from that assumption, applying such theorems already proved as are applicable to the diagram.

3. A consequence that contradicts some known truth shows the theorem to be false, and we can proceed to prove it so by a direct proof.

4. On arriving at a consequence known to be true, take this as the basis of a synthetic proof, retracing the steps by which this consequence was reached.

Given, for example, the following theorem :

153. THEOREM. *The bisectors of the angles of a triangle meet in a point.*



Given : In triangle ABC , AD , BE , CF , the bisectors of angles A , B , C , respectively ;

To Prove : AD , BE , CF , have a common point.

ANALYSIS. Let O be the point common to the three bisectors AD , BE , CF ; then O must be equidistant from AB , AC , and BC (101). But this condition is satisfied if there is a point common to BE and CF , since that point must be equidistant from AB and BC , from BC and AC (101). Now, in order that BE and CF may meet in a point, we must have $\angle EBC + \angle FCB < \text{a straight angle}$ (114). But

these angles *are* less than a straight angle, since they are the halves of $\angle ABC$, $\angle ACB$, respectively, which are less than a straight angle (84). Hence

SYNTHESIS. Since $\angle B + \angle C < \text{a st. } \angle$, (84)

$$\angle EBC + \angle FCB < \text{a st. } \angle,$$

$\therefore BE, CF$, will meet in a point O . (114)

Now, since O is in the bisector BE of $\angle ABC$,

O is equidistant from BA and BC ; }

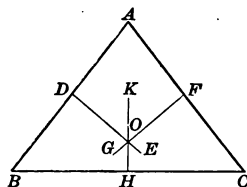
similarly O is equidistant from CA and BC , } (101')

$\therefore O$ is in the bisector AD of $\angle BAC$, Q.E.D. (101'')

(since it is equidistant from AB and AC .)

154. SCHOLIUM. The point of intersection of the bisectors of any two angles of a triangle is equidistant from all the sides.

155. THEOREM. *The perpendiculars at the mid points of the sides of a triangle meet in a point.*



Given: In triangle ABC , DE, FG, HK , perpendiculars to AB, AC, BC , at their respective mid points;

To Prove: DE, FG, HK , have a common point.

The analysis and synthesis of this theorem are so similar to those of the preceding theorem that they may be fairly left for the student to supply, with the remark that the analysis depends upon Prop. XXIII. and its first Cor. much as that of the preceding theorem depends upon Prop. XXV.

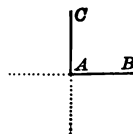
156. SCHOLIUM. The point of intersection of any two of the perpendiculars at the mid points of the sides of a triangle is equidistant from all the vertices.

157. DEFINITIONS. In any triangle, the perpendicular from a vertex to the opposite side is called the *altitude* to that side; the line joining a vertex with the mid point of the opposite side is called the *median* to that side; the sum of the sides is called the *perimeter*. In an isosceles triangle, the equal sides may be referred to as the *arms*, and the other side as the *base*.

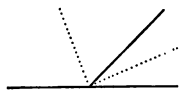
EXERCISES.

THEOREMS.

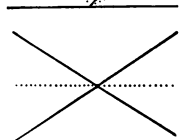
112. If the sides of a right angle BAC are produced through A , the new angles thus formed are all right angles. (30, 50)



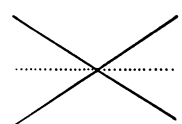
113. The bisectors of adjacent supplementary angles are perpendicular to each other. (46)



114. Conversely, if the bisectors of two adjacent angles are perpendicular to each other, those angles are supplementary. (46)



115. The bisectors of vertical angles are in the same straight line. (49)

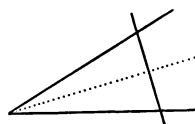


116. If a line bisect one of two vertical angles, it bisects the other also. (50)



117. The intercepts bisecting the base angles of an isosceles triangle are equal. (63)

118. Any intercept drawn perpendicular to the bisector of an angle cuts off equal parts from the sides. (63)



119. The altitudes to the arms of an isosceles triangle are equal. (63)

120. The bisectors of the base angles of an isosceles triangle form, if they meet, an isosceles triangle. (65)

121. The medians to the arms of an isosceles triangle are equal. (66)

122. Enunciate and prove the converse of Exercise 118. (66)

123. Enunciate and prove the converse of Exercise 120. (66)

124. Lines drawn from the extremities of the base of an isosceles triangle to points equally distant from the vertex are equal, and divide the arms into parts that are mutually equal. (66)

125. Lines drawn from the vertex of an isosceles triangle to points in the base equally distant from its extremities are equal. (66)

126. If equal distances from the vertices of an equilateral triangle be laid off in the same order, the lines joining these points form an equilateral triangle. (66)

127. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram. (66)

128. If a quadrilateral has two adjacent sides equal and making equal angles with the other two sides, these are also equal. (68)

129. An equilateral triangle is equiangular; and conversely. (68, 65)

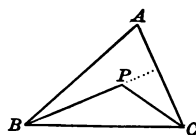
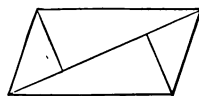
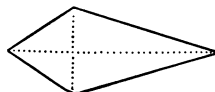
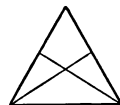
130. Prove that in the diagram for Exercise 128, the diagonals divide the figure into pairs of equal triangles. (69)

131. The perpendiculars to a diagonal of a parallelogram from the opposite vertices are equal. (73)

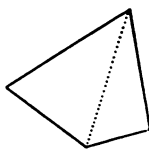
132. The diagonals of a square or rhombus bisect each other at right angles, and bisect the angles whose vertices they join. (74)

133. If from any point P in $\triangle ABC$, PB , PC , be drawn, then $\angle P > \angle A$, but

$$PB + PC < AB + AC. \quad (83, 88)$$

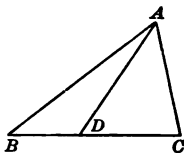


134. If in a quadrilateral the greatest side is opposite the least, each angle including the least side is greater than the opposite angle. (86, 88)

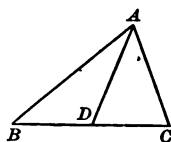


135. The sum of the lines drawn from any point in a triangle to the vertices is less than the sum, but greater than the semisum, of the sides. (88)

136. In a triangle ABC , if AB is not less than AC , any intercept AD between A and BC is less than AB . (87)



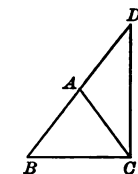
137. In a triangle ABC , a median AD is drawn to BC . Show that the angle ADB is obtuse, right, or acute, according as $AB >$, $=$, or $<$ AC . (91)



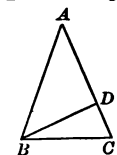
138. The median to any side of a triangle is less than the semisum of the other two sides, but greater than the semidifference of their sum and the third side. (88)



139. A line drawn from any point in the bisector of an angle, parallel to one side, meets the other in a point equidistant from the vertex and the point. (110)



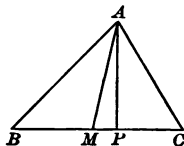
140. If AB , an arm of the isosceles triangle ABC , be produced so that $AD = AB$, then will the line joining DC be perpendicular to BC . (120)



141. If BD is the altitude upon AC , one of the arms of the isosceles triangle ABC , then the angle DBC is equal to one half the vertical angle. (120)



142. The sum of the medians to the sides of a triangle is less than the sum, but greater than the semisum, of the sides. (Exercise 138)



143. If AP is the altitude, and AM the bisector, from A to BC , then

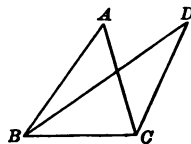
$$\angle PAM = \frac{1}{2}(\angle C - \angle B). \quad (120)$$

144. The median to the hypotenuse of a right triangle is equal to one half the hypotenuse. (120)

145. If the bisectors of two angles of an equilateral triangle meet, and, from the point of meeting, lines be drawn parallel to any two sides, these lines will trisect the third side. (120)

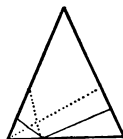
146. The bisector of an exterior angle at the vertex of an isosceles triangle is parallel to the base. (122)

147. If, in a triangle ABC , the bisectors of an interior angle at B and of an exterior angle at C meet in D , then $\angle D = \frac{1}{2} \angle A$.

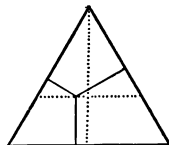


148. The angle formed by the bisectors of any two consecutive angles of a quadrilateral is equal to the sum of the other two angles. (125)

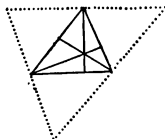
149. The sum of the distances of any point in the base of an isosceles triangle from the arms, is constant; i.e., is always equal to the altitude upon an arm. (143)



150. The sum of the distances of any point within an equilateral triangle from the sides is constant; i.e., is equal to an altitude. (Exercise 148)



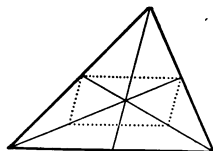
151. The lines joining the mid points of the sides of a triangle divide it into four equal triangles. (152)



152. The three altitudes of a triangle have a common point. (141, 155)

153. The lines joining the mid points of adjacent sides of any quadrilateral, form a parallelogram the sum of whose sides is equal to that of the diagonals of the quadrilateral. (147)

154. The three medians of a triangle have a common point which cuts off one third of each median. (147)



155. If the exterior angles of a triangle are bisected, the three exterior triangles formed on the sides of the original triangle are equiangular. (147)

156. The vertices of all right triangles having a common base as hypotenuse, lie in the same circumference. (Exercise 144)

BOOK II.

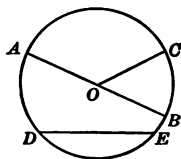
THE CIRCLE. LOCI. PROBLEMS.



ELEMENTARY PROPERTIES.

158. A *circle* is a plane figure bounded by a curved line such that a certain point within, called the *center*, is equidistant from every point of the curve.

159. The *circumference* of a circle is the curve that bounds it; as *ADEBC*.



160. An *arc* is any part of a circumference; as *AD* or *ACE*.

161. A *radius* is any straight line drawn from center to circumference; as *OA*.

162. Cor. *All radii of the same circle are equal* (158). Also, a point is *within, on, or without* a circumference according as its distance from the center is *less than, equal to, or greater than* a radius.

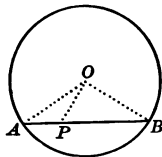
163. A *chord* is a straight line joining the extremities of an arc; as *DE*. The chord is said to *subtend* its arc.

164. A *diameter* is a chord that passes through the center; as *AB*.

165. Cor. *All diameters of the same circle are equal*. For each is double the radius of that circle.

PROPOSITION I. THEOREM.

166. *The straight line terminating in any two points in a circumference lies wholly within the circle.*



Given: A straight line, AB , terminating in two points in the circumference whose center is O ;

To Prove: AB lies within that circumference.

Take any point P between A and B , and join OA , OB , OP .

$$\text{Since } OA = OB, \quad (162)$$

$$\angle B = \angle A. \quad (68)$$

$$\text{But } \angle A < \text{ext. } \angle OPB, \quad (83)$$

$$\therefore \angle B < \angle OPB,$$

$$\therefore OP < OB, \quad (87)$$

i.e., P , which is any point between A and B , lies within the circle. Q.E.D.

167. COR. *A straight line can meet a circumference in not more than two points.*

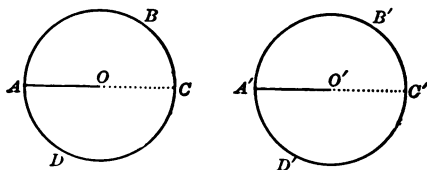
EXERCISE 157. Show that a circle cannot have more than one center. State the axioms upon which your proof depends.

158. A straight line will cut a circle, or lie entirely without it, according as its distance from the center is less than, or greater than, the radius of the circle.

159. If a straight line could meet a circumference in three points, how many equal straight lines could be drawn from the center to that line?

PROPOSITION II. THEOREM.

168. *Circles having equal radii are equal.*



Given: Two circles, ABD , $A'B'D'$, having radius OA equal to radius $O'A'$;

To Prove: Circle ABD is equal to circle $A'B'D'$.

Place $\odot ABD$ upon $\odot A'B'D'$, so that $OA \neq O'A'$.

Then, not only will A coincide with A' , but also circumference ABD with circumference $A'B'D'$, since all points of both circumferences are equally distant from the coinciding centers O and O' (Hyp. and 158). Hence,

since circumf. $ABD \neq$ circumf. $A'B'D'$,

$$\odot ABD = \odot A'B'D'. \quad \text{Q.E.D.} \quad (14)$$

169. COR. *A diameter bisects the circle and its circumference.*

For producing AO , $A'O'$, to form the diameters AC , $A'C'$; since the part ABC may be made to coincide, as above, either with $A'B'C'$ or $A'D'C'$, these parts are equal, as are also their bounding arcs. (Ax. 1)

170. DEFINITION. A half circle is called a *semicircle*; a half circumference, a *semicircumference*.

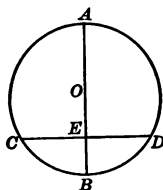
EXERCISE 160. How many points are necessary to determine the magnitude and position of a circle in a given plane?

161. Can two circles have a common center without coinciding?

CHORDS.

PROPOSITION III. THEOREM.

171. *If a chord is perpendicular to another chord at its mid point, the first chord is a diameter.*



Given: Chord AB perpendicular to chord CD at its mid point E ;

To Prove: AB is a diameter.

Since AB is \perp to CD at its mid point, (Hyp.)

AB contains all the points that are equidistant from C and D ; (96)

\therefore the center of $\odot ACD$ lies in AB , (158)

$\therefore AB$ is a diam. of $\odot ACD$. Q.E.D. (164)

172. COR. *A radius perpendicular to a chord bisects that chord.* (97)

EXERCISE 162. In the diagram for Prop. III., if AC , AD , be joined, then $AC = AD$.

163. The line drawn through the mid points of parallel chords in a circle, passes through the center.

164. If an isosceles triangle be constructed on any chord of a circle, its vertex will be in a diameter or a diameter produced.

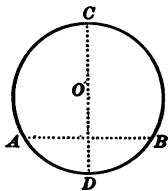
165. If two chords of a circle bisect each other, both are diameters.

166. Enunciate the converse of Prop. III. Is that converse always true?

167. If from any point within a circle two equal straight lines be drawn to the circumference, the bisector of the angle they form will pass through the center.

PROPOSITION IV. PROBLEM.

173. To find the center of a given circle.



Given: A circle ACB ;
Required: To find its center.

Join any two points A, B , in the circumf.

Draw $CD \perp$ to AB at its mid point. (95)

Bisect CD in O (80). O is the required center.

For chord CD is \perp to AB at its mid point, (Const.)

\therefore the center of $\odot ACB$ lies in CD ; (171)

and since $OC = OD$, (Const.)

O is the center of $\odot ACB$, Q.E.F.

(it being the mid point of the diam. CD .)

SCHOLIUM. In order to avoid repetition, it will usually be assumed, henceforth, that the center O of any circle, if not given, has been found by this construction.

EXERCISE 168. Show how to find the center of a circle, when an arc only of its circumference is given.

169. Apply Exercise 167 so as to find the center of a circle without the bisection of any chord.

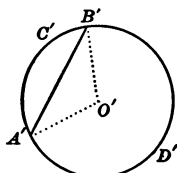
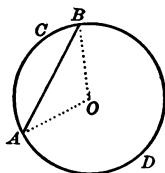
170. Apply Exercise 168 to effect the same purpose.

171. Find the direction in which lies the longest, and also the shortest, distance from a point within a circle to its circumference.

172. Find the same for a point without the circle.

PROPOSITION V. THEOREM.

174. *In the same circle, or in equal circles, equal arcs are subtended by equal chords; and conversely.*



1°. *Given:* In equal circles ADB , $A'D'B'$, AB , $A'B'$, chords of equal arcs ACB , $A'C'B'$;

To Prove: Chord AB is equal to chord $A'B'$.

O , O' , being the centers, place $\odot ADB$ upon $\odot A'D'B'$,

so that $O \neq O'$ and $A \neq A'$. Then

since arc $ACB = \text{arc } A'C'B'$, (Hyp.)

B will coincide with B' ,

\therefore chd. AB will coincide with, and equal, chd. $A'B'$. Q.E.D. (14)

2°. *Given:* In equal circles ADB , $A'D'B'$, equal chords AB , $A'B'$;

To Prove: Arc ACB is equal to arc $A'C'B'$.

Join OA , OB , $O'A'$, $O'B'$. Then

since $OA = O'A'$, $OB = O'B'$, and $AB = A'B'$, (Hyp.)

$$\triangle OAB = \triangle O'A'B', \quad (69)$$

$\therefore \odot ADB$ can be placed upon $\odot A'D'B'$, so that

$$\triangle OAB \neq \triangle O'A'B'; \quad (61)$$

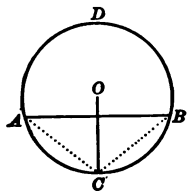
then arc $ACB \neq \text{arc } A'C'B'$,

$$\therefore \text{arc } ACB = \text{arc } A'C'B'. \quad \text{Q.E.D. (14)}$$

SCHOLIUM. If the arcs are in the same circle, the proof is similar to that above.

PROPOSITION VI. THEOREM.

175. *A radius bisecting a chord bisects also its subtended arc.*



Given : A radius OC bisecting chord AB ;

To Prove : Arc BCA is bisected in C .

Join CA and CB . Then

since OC is \perp to AB at its mid point, (Hyp.)

chd. $CA = \text{chd. } CB$, (96)

\therefore arc $CA = \text{arc } CB$. Q.E.D. (174)

176. COR. 1. *A radius bisecting an arc bisects its chord at right angles.* (75)

For the extremities of the radius are equally distant from those of the chord.

177. COR. 2. *A radius bisecting a chord or its subtended arc is perpendicular to the chord.* (75 and 176)

EXERCISE 173. In the diagram for Prop. V., if the bisector of angle AOB be drawn, it will bisect arc ACB and also arc ADB .

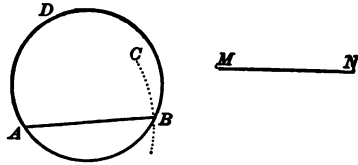
174. In the same diagram, if the points where the bisector cuts the circumference be joined with A and B , the chords drawn from each point will be equal.

175. In the diagram for Prop. VI., if diameters from A and B meet the circumference in E and F , the line joining EF will be parallel to AB .

176. Two intersecting circles cannot have the same center.

PROPOSITION VII. PROBLEM.

178. In a given circle, to place a chord equal to a given straight line not greater than a diameter.



Given: A straight line MN not greater than the diameter of a circle ADB ;

Required: To place in ADB a chord equal to MN .

From A , any desired point in the circumference, as center, with radius equal to MN , describe an arc CB , cutting the circumference in B .

Join AB . AB is the required chord.

Since AB is a radius of arc BC , and $AB = MN$, (Const.) a chd. AB equal to MN has been placed in $\odot ADB$. Q.E.F.

179. SCHOLIUM. By this construction, any given arc may be laid off on a circumference having a radius equal to that of the arc.

180. DEFINITION. An arc is called a *major* or a *minor* arc, according as it is *greater* or *less* than a semicircumference.

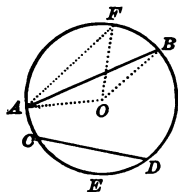
EXERCISE 177. In the diagram for Prop. VII., if arc BC be produced to meet the circumference in D , and BD be drawn, the diameter drawn from A will bisect BD .

178. In the same diagram, place a chord that is equal and parallel to AB .

179. In the same diagram, place a chord that is equal and perpendicular to AB .

PROPOSITION VIII. THEOREM.

181. *In the same circle, or in equal circles, the greater of two minor arcs is subtended by a greater chord; and conversely.*



1°. *Given:* In circle ADB , arc AFB greater than arc CED ;
To Prove: Chord AB is greater than chord CD .

From A draw chd. $AF = CD$ (178), and join O , the center, with A , B , F .

Since chd. $AF =$ chd. CD , (Const.)

arc $AF =$ arc CED (174'') and $<$ arc AFB , (Hyp.)

$\therefore F$ falls between A and B , and OF between OA and OB ,

$\therefore \angle AOB > \angle AOF$. (Ax. 8)

Now $OA, OB = OA, OF$, respectively, (162)

$\therefore AB > AF$ or CD . Q.E.D. (90)

2°. *Given:* In circle ADB , chord AB greater than chord CD ;
To Prove: Arc AFB is greater than arc CED .

With the same construction as in 1°,

since $OA, OB = OA, OF$, respectively, (162)

but $AB > AF$, (Hyp.)

$\angle AOB > \angle AOF$, (90)

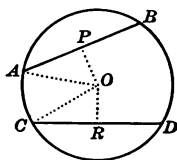
$\therefore F$ falls between A and B ,

\therefore arc $AFB >$ arc AF or arc CD . Q.E.D. (Ax. 8)

SCHOLIUM. If the arcs are in equal circles, the proof is similar to that above; and, in general, the demonstration of a theorem concerning arcs, etc., in *equal* circles would be similar to that for arcs in the *same* circle, and *vice versa*.

PROPOSITION IX. THEOREM.

182. *In the same circle, or in equal circles, equal chords are equally distant from the center; and conversely.*



1°. *Given*: In circle ADB , chord AB equal to chord CD ;

To Prove: AB and CD are equally distant from O , the center.

Draw $OP \perp$ to AB , $OR \perp$ to CD , and join OA , OC .

Since OP is \perp to AB , and OR is \perp to CD , (Const.)

$$AP = \frac{1}{2} AB, \text{ and } CR = \frac{1}{2} CD, \quad (172)$$

$$\therefore AP = CR \text{ (Ax. 7), and } OA = OC, \quad (162)$$

$$\therefore \text{rt. } \triangle OAP = \text{rt. } \triangle OCR, \quad (72)$$

$$\therefore OP = OR. \quad \text{Q.E.D.} \quad (70)$$

2°. *Given*: In circle ADB , chords AB , CD , equally distant from O , the center;

To Prove: $AB = CD$.

With the same construction as in 1°,

$$\text{since } OP = OR \text{ (Hyp.), and } OA = OC, \quad (162)$$

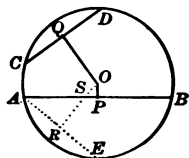
$$\text{rt. } \triangle OAP = \text{rt. } \triangle OCR, \quad (72)$$

$$\therefore AP = CR, \quad (70)$$

$$\therefore 2AP \text{ or } AB = 2CR \text{ or } CD. \text{ Q.E.D. (Ax. 6)}$$

PROPOSITION X. THEOREM.

183. *In the same circle, or in equal circles, a greater chord is nearer the center; and conversely.*



1°. *Given: In circle ADB, OP perpendicular to chord AB, OQ perpendicular to chord CD, and AB greater than CD;*

To Prove: OP is less than OQ.

Since $AB > CD$ (Hyp.), arc $AEB > \text{arc } CD$. (181")

On arc AEB lay off arc $AE = \text{arc } CD$. (178)

Join AE , and draw $OR \perp$ to AE ; (92)

then $AE = CD$ (174'), and $OR = OQ$. (182')

Since AB lies between O and AE , (Const.)

OR must cut AB in some point S ,

$\therefore OS < OR$. (Ax. 8)

But $OP < OS$, (93)

$\therefore OP < OR$ or OQ . Q.E.D. (a.f.)

2°. *Given: In circle ADB, OP perpendicular to chord AB, OQ perpendicular to chord CD, and OP less than OQ;*

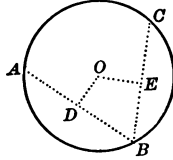
To Prove: Chord AB is greater than chord CD.

Chord AB cannot be equal to chord CD , for then OP would be equal to OQ (182'); nor can AB be less than CD , for then OP would be greater than OQ (1°). Hence, since AB can be neither equal to nor less than CD , it must be greater than CD .

184. COR. *A diameter is greater than any chord not passing through the center.*

PROPOSITION XI. THEOREM.

185. *Through any three given points not in the same straight line one circumference can be described, and only one.*



Given : Three points A, B, C , not in the same straight line ;

To Prove : One circumference can be described through A, B , and C .

Join AB, BC , and at D, E , the mid points of AB, BC , respectively, draw $DO \perp$ to AB , and $EO \perp$ to BC .* (80)

$\therefore DO, EO$, are \perp to AB, BC , resp., at their mid points, (Const.)

DO, EO , will meet at a point O equidistant from A, B , and C ; (156)

\therefore the circumference described from O as center, with radius OA , will pass through A, B , and C . Q.E.D. (162)

Moreover, there can be but one such circumference. For the center of any circumference passing through A, B , and C , must lie both in DO and in EO (97), which lines can have but one common point.

186. COR. *Two circumferences can intersect in not more than two points.*

For if they could intersect in three points, there would be two different circumferences passing through the same three points, which is impossible.

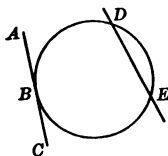
187. SCHOLIUM. One circumference, and only one, can be described through the vertices of a given triangle.

* The construction by which we find the mid point of a line, evidently also gives the perpendicular at that point.

TANGENTS AND SECANTS.

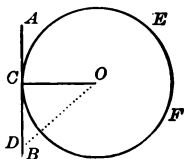
188. A *tangent* is a straight line that touches a circumference in one point only; as ABC . The point B in which the tangent touches the circle is called the *point of contact*.

189. A *secant* is a straight line that cuts a circumference in two points; as DE .



PROPOSITION XII. THEOREM.

190. A straight line perpendicular to a radius at its extremity is a tangent to the circle.



Given: AB perpendicular to OC , a radius of circle CEF , at its extremity C ;

To Prove: AB is a tangent to circle CEF .

Join O with D , any point in AB except C . Then

since OC is \perp to AB , (Hyp.)

OD is oblique to AB , (51)

$\therefore OD > OC$, (93)

$\therefore D$ lies without the circle CEF , (162)

$\therefore AB$ is tangent to CEF at C , Q.E.D. (188)

(since C is the only point in AB not without CEF .)

191. COR. 1. A tangent is perpendicular to the radius drawn to the point of contact.

For OC must be less than any other line drawn from O to AB (162).

192. COR. 2. *A perpendicular to a tangent at the point of contact passes through the center of the circle.*

For otherwise there could be two perpendiculars to the tangent at that point (191).

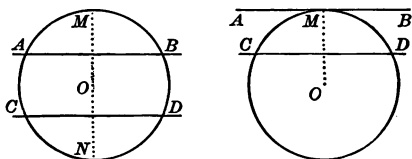
193. COR. 3. *A perpendicular from the center to a tangent meets it at the point of contact (192).*

194. COR. 4. *At a given point of contact there can be but one tangent.*



PROPOSITION XIII. THEOREM.

195. *Parallels intercept equal arcs on a circumference.*



Given: Two parallels AB , CD , cutting or touching the circumference whose center is O ;

To Prove: These parallels intercept equal arcs.

1°. Let AB , CD , be secants.

Through O , the center, draw the diameter $MN \perp$ to AB .

Since AB is \parallel to CD (Hyp.), and MN is \perp to AB , (Const.)

$$MN \text{ is } \perp \text{ to } CD, \quad (107)$$

$$\therefore \text{arc } MA = \text{arc } MB, \text{ and } \text{arc } MC = \text{arc } MD, \quad (175)$$

$$\therefore \text{arc } AC = \text{arc } BD. \quad \text{Q.E.D.} \quad (\text{Ax. 3})$$

2°. Let AB be tangent at M , and CD a secant.

$$\text{Draw } OM. \text{ Then } OM \text{ is } \perp \text{ to } AB \quad (191)$$

$$\text{and also to its parallel } CD; \quad (107)$$

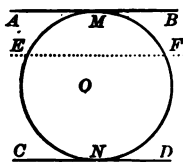
$$\therefore \text{arc } MC = \text{arc } MD. \quad \text{Q.E.D.} \quad (175)$$

3°. Let AB , CD , be tangents at M , N , respectively.

Draw a secant $EF \parallel$ to AB ; it will also be \parallel to CD . (108)

Since arc $ME =$ arc MF , and arc $NE =$ arc NF , (2°)

arc $MEN =$ arc MFN . Q.E.D. (Ax. 2)



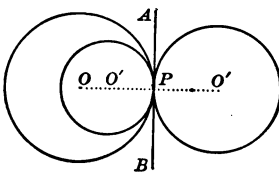
196. DEFINITION. The line joining the centers of two circles is their *line of centers*; the distance between their centers is their *central distance*.

197. Two circles are tangent to each other when both are tangent to the same straight line at the same point.

198. A circle tangent to another is tangent to it *internally* or *externally* according as it lies within or without the other circle.

PROPOSITION XIV. THEOREM.

199. If two circles are tangent to each other, their line of centers passes through the point of contact.



Given: Circles with centers O , O' , tangent at P in AB ;

To Prove: P lies in the line joining OO' .

Through P draw a perpendicular to AB . (95)

Since AB is tangent to both circles at P , (Hyp.)

the \perp to AB at P passes through O and O' , (192)

i.e., P is in the line of centers OO' . Q.E.D.

EXERCISES.

QUESTIONS.

180. Can you find the center of a circle without bisecting any straight line?

181. If a chord of one circle is equal to a chord of another, are these circles necessarily equal? If not, what other condition must be fulfilled that the circles may be equal?

182. In the diagram for Prop. VII., if MN were equal to a diameter, through what point would AB necessarily pass?

183. In equal circles, which is subtended by the greater chord, the greater or the less of two major arcs?

184. Can a tangent be drawn to a circle from a point within it?

185. How may a tangent be drawn at a given point in a circumference when the center is not known?

186. How many points determine a circumference?

187. Is there any limit to the number of circles that may be described through two given points?

188. In what case is it impossible to pass a circle through three given points?

189. Can a circumference always be described through the angular points of a rectangle?

190. How many circles of equal radii may touch a given straight line at a given point?

191. How many circles of any radii may touch a given straight line at a given point?

THEOREMS.

192. A circle is wholly without or within another circle, according as their central distance is greater than the sum, or less than the difference, of their radii. (162)

193. A circle is tangent to another externally or internally, according as their central distance is equal to the sum or the difference of their radii. (199)

194. Two circles intersect if their central distance is less than the sum, and greater than the difference, of their radii. (88, 89),

195. If two circles intersect, their line of centers is perpendicular to their common chord at its mid point. (172)

196. The center is the only point from which more than two equal lines can be drawn to the circumference. (97, 171)

197. If two equal chords intersect, their segments are severally equal. (174, 69)

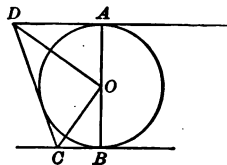
198. If through any point in a radius two chords be drawn making equal oblique angles with it, these chords are equal. (182'')

199. The chord drawn through any point of a radius perpendicular to it is the least chord that can be drawn through that point. (183)

200. The tangents drawn to a circle from any point without it are equal, and make equal angles with the line joining the point with the center. (192)

201. The sums of the opposite sides of a quadrilateral described about a circle are equal. (Exercise 200)

202. Tangents AD , BC , at the extremities of a diameter AB , meet another tangent CD in C and D ; join OC , OD . (1) $\angle COD$ is a right angle; (2) $CD = AD + BC$. (Exercise 200)



203. If through the points of intersection of two circumferences, parallels be drawn to meet the circumferences, these parallels will be equal. (195, 174)

204. If two tangents drawn to a circle from the same point, intercept between them a third tangent touching any point of the intercepted arc, the perimeter of the triangle formed by the three tangents is constant. (Exercise 200)

205. If a circle is inscribed in a triangle, the distances of the vertex of any angle to the points of contact of its sides are equal to the semiperimeter of the triangle less the side opposite the angle. (Exercise 200)

206. If a circle is inscribed in a trapezoid that has equal angles at its base, each nonparallel side is equal to half the sum of the parallel sides. (Exercise 200)

207. If two circles are each tangent to a pair of parallel lines and also to a transverse intercept between the parallels, the intercept is equal to the central distance of the circles. (Exercise 200)

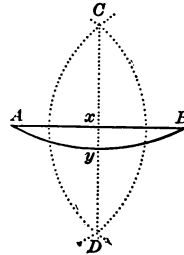
CONSTRUCTIONS.

Certain problems of construction have already been given as occasion arose for their use. The form in which they were presented was not, however, always the simplest possible, but such as the means of demonstration at that stage admitted. Those already given are, accordingly, presented again in a simplified form, with the directions for their construction worded as briefly as possible. The demonstrations the student should complete for himself with the aid of the diagrams and references.

200. *To bisect a given straight line, or arc, AB .*

From A and B , with any radius greater than half the distance AB , describe arcs intersecting in C and D .

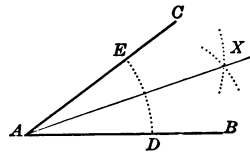
The line CD bisects the line AB in x , and the arc AB in y (74, 175).



201. *To bisect a given angle BAC .*

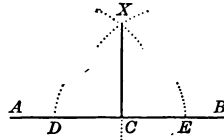
From A , with any radius, describe an arc cutting AB , AC , in D , E , respectively. From D and E , with the same radius, describe arcs intersecting in X .

The line AX is the bisector of $\angle BAC$ (101").



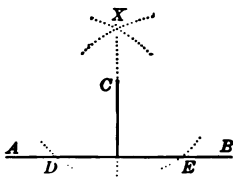
202. *To draw a perpendicular to a given line AB through a point C in or without AB .*

From C , with any suitable radius, describe an arc cutting AB in D and E . From D and E , with equal radii of suitable length, describe arcs intersecting in X .



The line CX is perpendicular to AB (74).

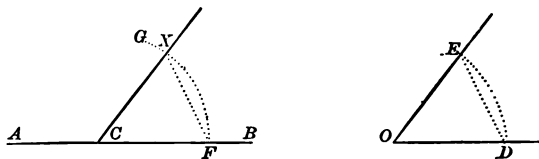
It will be seen that the simplification in these cases arises from the use of arcs of any convenient radius instead of



whole circumferences of prescribed radius, and in the omission of construction lines needed only for demonstration.

PROPOSITION XV. PROBLEM.

203. *At a point C, in a given straight line AB, to construct an angle equal to a given angle O.*



Given: A point C in a straight line AB , and an angle O ;

Required: To make at C an angle equal to angle O .

From O as center, with any radius, describe an arc cutting the sides of $\angle O$ in D and E .

From C as center, with the same radius, describe an arc FG , and on FG set off an arc FX equal to arc DE . (179)

Join CX . FCX is the required angle.

Join FX . Then

since FX and DE are equal arcs of equal circles, (Const.)

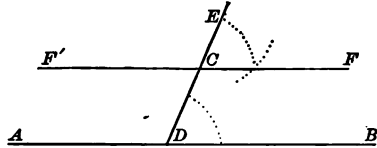
chord $FX = \text{chord } DE$; (174')

also CF and $CX = OD$ and OE , resp. (Const.)

$\therefore \triangle FCX = \triangle DOE$. $\therefore \angle C = \angle O$. Q.E.F.

PROPOSITION XVI. PROBLEM.

204. *Through a given point to draw a parallel to a given line.*



Given: A straight line AB and a point C ;

Required: To draw through C a parallel to AB .

Through C draw ECD , making any $\angle D$ with AB .

At the point C in ED make $\angle ECF = \angle D$. (203)

$F'CF$ is the required parallel.

For since $\angle C = \angle D$, (Const.)

$F'CF$ is \parallel to AB . Q.E.F. (112')

EXERCISE 208. The common chord of two intersecting circles may be a diameter of one, but not of both.

209. The chords joining the extremities of equal parallel chords of the same circle form with them a rectangle.

210. In the diagram for Prop. IX., if BA , DC , be produced to meet in Q , the line joining OQ will bisect $\angle BQD$.

211. Hence show that, in the diagram for Prop. VIII., if AC , DF , be joined, AC will be parallel to DF .

212. In the diagram for Prop. XI., show that $\angle O$ is the supplement of $\angle B$.

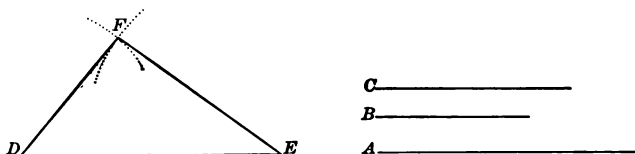
213. In the same diagram, if OD , OE , be produced to meet the circumference in F and G , then will arc $FBG = \frac{1}{2}(\text{arc } AB + \text{arc } BC)$.

214. In the same diagram, if HK be tangent to the circle at B , then will $\angle HBD + \angle KBE$ be equal to $\angle O$.

215. Tangents at the extremities of a diameter are parallel, and conversely. If two radii are at right angles to each other, the tangents at their extremities are perpendicular to each other.

PROPOSITION XVII. PROBLEM.

205. To construct a triangle having its sides respectively equal to three given straight lines, each less than the sum of the other two.



Given: Three straight lines, A , B , C , each less than the sum of the other two;

Required: To construct a triangle having sides equal to A , B , C , respectively.

As base, draw a line $DE = A$ say.

From D as center, with radius equal to B , draw an arc.

From E as center, with radius equal to C , draw another arc.

Since $DF + FE > DE$, the central distance, (Hyp.)
the arcs will intersect in some point F .

Join DF , EF . DEF is the required \triangle ;

its sides being equal to A , B , C , resp. Q.E.F. (Const.)

SCHOLIUM. If the given lines are all equal, the triangle will be *equilateral*; if two are equal, the triangle will be *isosceles*.

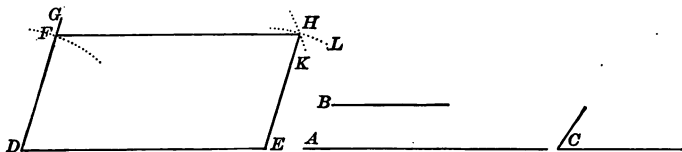
EXERCISE 216. The angle formed by two tangents drawn to a circle from the same point, is supplementary to that formed by the radii to the points of contact.

217. Show that the angle formed by those radii is bisected by the line joining its vertex with the center of the circle.

218. If a tangent and a secant be drawn to a circle from the same point, the angle formed by them will be equal to, or supplementary to, that formed by radii perpendicular to them, according as they do or do not lie on the same side of the center.

PROPOSITION XVIII. PROBLEM.

206. To construct a parallelogram with adjacent sides respectively equal to two given lines, and including a given angle.



Given: Two straight lines A and B , and an angle C ;

Required: To construct a parallelogram with two sides equal to A , B , respectively, including an angle C .

Draw a line $DE = A$, and at D make $\angle GDE = \angle C$. (203)

On DG lay off $DF = B$.

From F as center, with radius equal to DE , describe an arc HK .

From E as center, with radius equal to DF , describe an arc HL .

Let the arcs intersect in H , and join FH , EH ;

DH is the parallelogram required.

Since $FH = DE$, $EH = DF$, and $\angle D = \angle C$, (Const.)

DH is a parallelogram (141), with sides $= A$, B , resp., and with an angle $= \angle C$. Q.E.F.

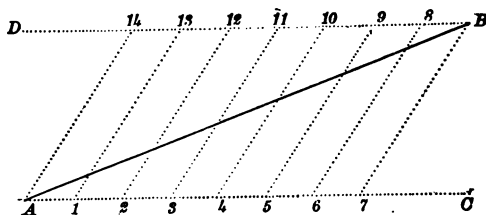
SCHOLIUM. When the given angle is a right angle, this construction gives a *rectangle*. When the given sides are equal, the construction gives a *square* or *rhombus*, according as the given angle is or is not a right angle.

EXERCISE 219. Construct a rectangle having one side double the other.

220. Construct a square and circumscribe a circle about it.

PROPOSITION XIX. PROBLEM.

207. To divide a given straight line into any number of equal parts.



Given : A straight line AB ;

Required : To divide AB into n equal parts.

Let $n = 7$. Draw AC , making any convenient \angle with AB , and draw $BD \parallel$ to AC . (204)

On AC lay off 7 equal parts, $A-1, \dots 6-7$, of any convenient length.

On BD lay off 7 equal parts, $B-8, \dots 13-14$, of the same length as in AC .

Join $A-14, 1-13, \dots 7-B$. AB is divided into 7 equal parts.

Since $A-1$ is equal and parallel to $13-14$, (Const.)

$A-14$ is \parallel to $1-13$. (142)

Similarly, all the lines $2-12, 3-11, \dots 7-B$, are \parallel to $A-14$.

Since these \parallel 's cut off 7 equal intercepts on $A-7$, (Const.)

these \parallel 's cut off 7 equal intercepts on AB , (151)

i.e., AB is divided into 7 equal parts. Q.E.F.

We proceed in the same way if n is any other integer.

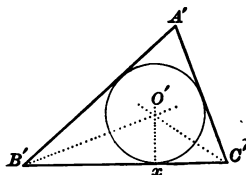
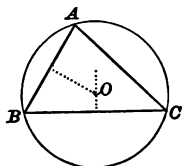
SCHOLIUM. In practice, it will be sufficient to lay off only one part on BD , as $B-8$; then by joining 8 with the corresponding division on AC , as 6, we cut off one of the required equal parts on AB ; then lay off the others.

208. DEFINITION. A circle is said to *circumscribe* a polygon when its circumference passes through each of the angular points of the polygon, which is then said to be *inscribed* in the circle.

209. DEFINITION. A circle is said to be *inscribed* in a polygon when it touches each of the sides of the polygon, which is then said to be *circumscribed* about the circle.

PROPOSITION XX. PROBLEM.

210. *To circumscribe a circle about, or to inscribe a circle in, a given triangle.*



Given: Two triangles ABC , $A'B'C'$.

Required: 1°, To circumscribe a circle about ABC ;

2°, To inscribe a circle in $A'B'C'$.

1°. When, by the construction given in Prop. XI., a circumference has been passed through the three points A , B , C , we have a circle circumscribing $\triangle ABC$ (208). Q.E.F.

2°. When, by the construction given in Art. 153, we have found a point O' that is equidistant from $A'B'$, $A'C'$, $B'C'$, then the circle described from O' as center, with any of the equal perpendiculars from O' to the sides as radius, will touch each of the sides, and thus be inscribed in $\triangle A'B'C'$ (209). Q.E.F.

PLANE LOCI.

211. The nature of loci becomes clear when we conceive a line as being the path generated by a point moving according to some specified condition. If the point move so as to be always at a given distance from a given fixed point, the line thus traced will be the locus of all points that are at the given distance from the given point. Hence,

A *locus* in a plane is the line, or lines, every point in which satisfies certain conditions fulfilled by no point not in that line or lines.

The following theorems upon the subject are merely the enunciation from a special point of view of certain theorems already proved. Though the proofs are here merely indicated by references, the student would find it useful to frame a demonstration of each of these theorems, keeping in mind that, in order that a line be a locus,

- (1) *Every point in the line must satisfy the given conditions ;*
- (2) *No point not in the line should satisfy them.*

212. THEOREM. *The locus of all the points situated at a given distance from a given point is the circumference described from that point as center with a radius equal to the given distance. (162)*

213. THEOREM. *The locus of all the points that are equidistant from two given points is the perpendicular at the mid point of the line joining those points. (20, 96)*

214. THEOREM. *The locus of all the points situated at a given distance from a given straight line consists of the two parallels drawn at the given distance from the given line, one on each side. (139)*

215. THEOREM. *The locus of all the points that are equidistant from two given parallels is the parallel through a point equidistant from both. (139)*

216. THEOREM. *The locus of all the points that are equidistant from two intersecting lines consists of the bisectors of the vertical angles formed by those lines. (101)*

217. THEOREM. *The locus of the mid points of parallel chords is the diameter perpendicular to those chords.* (172)

218. THEOREM. *The locus of the mid points of equal chords is the circumference concentric with the given circumference, and having a radius equal to the common distance of the chords.* (182)

Familiarity with these and similar theorems is of great importance, especially in the solution of problems, which very often depends upon the intersection of loci. The following easy exercises will be found useful as an introduction to the application of loci to the solution of problems.

EXERCISES.

LOCI.

NOTE.—The line AB referred to in these exercises is supposed to be a straight line of indefinite—that is, of any requisite—length.

221. In the line AB find a point that shall be at a given distance CD from a given point P . State the conditions under which no such point can be found; one point; more than one.

222. In the line AB find a point that shall be equidistant from two given points P and Q . State the conditions, etc., as in Exercise 221.

223. In the line AB find a point that shall be at a given distance CD from a given line EF . State the conditions, etc.

224. In the line AB find a point that shall be equidistant from two given parallels CD , EF . State the conditions, etc.

225. In the line AB find a point that shall be equidistant from two intersecting lines CD , EF . State the conditions, etc.

226. In a given circumference ABM find a point that shall be at a given distance CD from a given point P . State the conditions, etc.

227. In a given circumference ABM find a point that shall be equidistant from two given points P and Q . State the conditions, etc.

228. In a given circumference ABM find a point that shall be at a given distance CD from a given line EF . State the conditions, etc.

229. In a given circumference ABM find a point that shall be equidistant from two given parallels CD , EF . State the conditions, etc.

230. In a given circumference ABM find a point that shall be equidistant from two intersecting lines CD , EF . State the conditions, etc.

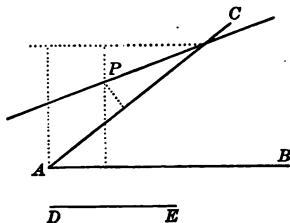
231. Find a point equidistant from two given points P and Q , and at a given distance CD from a given line AB . State the conditions, etc.

232. Find a point equidistant from two given points P and Q , and also equidistant from two parallels AB and CD . State the conditions, etc.

233. Find a point equidistant from two given points P and Q , and also equidistant from two intersecting lines AB and CD . State the conditions, etc.

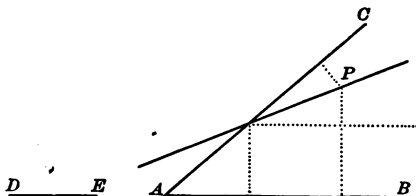
234. Within a given angle BAC find the point that is at a given distance DE from each of the sides.

235. Find a point that shall be at a given distance AB from a given point P , and at a given distance CD from a given point Q . State the conditions, etc.

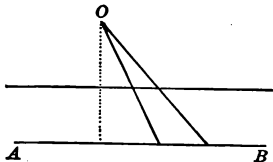


236. Find the locus of a point P such that the sum of its distances from the sides of a given angle BAC is always equal to a given line DE . (101)

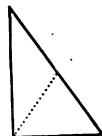
237. Find the locus of a point P such that the difference of its distances from the sides of a given angle BAC is always equal to a given line DE . (101)



238. Find the locus of the mid points of all the lines drawn from O to a line AB . (147)



239. Find the locus of the mid point of a ladder whose lower end is being pulled away from a wall. (144)



ANALYSIS OF PROBLEMS.

Most of the foregoing simple exercises are problems depending so plainly for solution upon the intersection of certain loci that no previous analysis is necessary. In general, the solution of a problem will be found to depend upon several problems or theorems already solved or proved, and the solution will be greatly facilitated by a previous analysis. From the nature of the case, no precise rules can be given, but the general course of procedure is expressed by the following rules.

1. *Construct a diagram in accordance with the statement of the problem, as if the required construction were effected.*

2. *Study the relations of the lines, angles, etc., in the diagram, so as to discover whether the assumed solution can be made to depend upon some known problem or theorem, especially those concerning loci.*

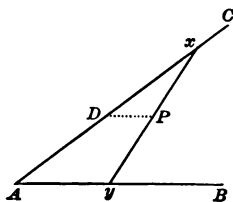
3. *If such dependence cannot be found by means of the original diagram, make such additions to it as the case may suggest, by joining points, drawing parallels or perpendiculars, etc., and proceed as in 2.*

4. *On discovering the dependence of the solution upon some known theorem or problem, make this the basis of a synthetic solution, proceeding in reverse order through the several steps of the previous analysis, till it is shown that the problem is solved. For example :*

219. PROBLEM. *Through a given point within an angle draw a straight line intercepted between the sides and bisected in the given point.*

ANALYSIS. 1. Let P be the given point within $\angle BAC$, and suppose xy drawn through P so that $Px = Py$.

2. The diagram as it stands is not suggestive.



3. Through P draw $PD \parallel$ to AB , to meet AC in D .
4. It is at once apparent that since PD is drawn through the mid point of xy and is \parallel to AB ,
 $\therefore D$ is the mid point of Ax . (147)

SYNTHESIS. Having thus found that the solution depends upon Prop. XLI. (147), we proceed as follows:

Through P draw $PD \parallel$ to AB to meet AC in D . (109)

In DC take $Dx = DA$, and through x draw xPy , meeting AB in y . xy is the required line.

Since D is the mid point of Ax ,
 and DP is \parallel to AB , the base of $\triangle xAy$, } (Const.)
 $\therefore xy$ is bisected in P . Q.E.F. (147)

The complete diagram required in the solution of a problem contains:

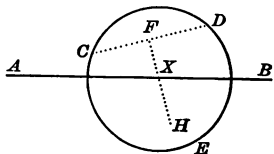
1. The *data*, or given figure; as point P in $\angle BAC$, above.
2. The *quæsitæ*, or things required; as the line xPy .
3. *Construction lines* employed as auxiliaries in the solution; as PD .

Of these, the data may conveniently be distinguished, as above, by the first letters of the alphabet; the quæsitæ, as far as requiring new letters, by the last letters; and the auxiliary constructions by dotted lines.

We give another example in order to show the application of the principle of intersecting loci to the solution of problems.

PROBLEM. Describe a circumference passing through two given points and having its center in a given straight line.

ANALYSIS. 1. Let AB be the given line, and C, D , the given points; and suppose circle CDE , having its center at X in AB , passes through C and D .



2. Since X is equidistant from C and D , it should be found in the locus of all points that are equidistant from C and D . Now that locus is (213) the perpendicular at the mid point of the line joining C and D ;

$\therefore X$ is found at the intersection of that locus with AB .

SYNTHESIS. Join CD . Draw the $\perp FH$ at the mid point of CD , and let FH intersect AB in X . From X as center, with a radius equal to the distance XC , describe the circumference CDE . CDE is the required circumference.

Since X is a point in the \perp at the mid point of CD ,

X is equidistant from C and D , (213)

\therefore a circumference passing through C will also pass through D , (164)

\therefore a circumference having its center in AB has been described through C and D . Q.E.F.

SCHOLIUM. The problem becomes impossible in a certain case. In what case?



EXERCISES.

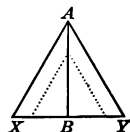
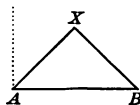
PROBLEMS.

240. Construct an isosceles triangle having its sides each double the length of the base.

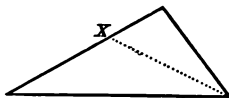
241. Upon a given base AB , construct a right isosceles triangle.

242. With a given line AB as diagonal, construct a square.

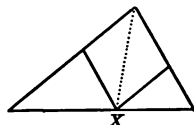
243. Construct an equilateral triangle having a given altitude AB .



244. In any side of a triangle, find the point which is equidistant from the other two sides.

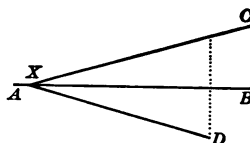


245. In any side of a triangle, find the point from which the lines drawn parallel to the other side are equal.

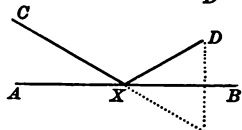


246. Trisect a given right angle.

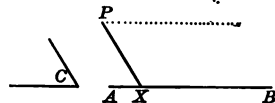
247. In a given line AB , find a point X such that the angle formed by lines drawn from X to two given points C, D , on opposite sides of AB , shall be bisected by AB .



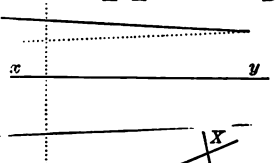
248. In a given line AB , find a point X such that the angles formed by lines drawn from X to two given points C, D , on the same side of AB , shall be equal. How may Exercises 247 and 248 be enunciated as one problem?



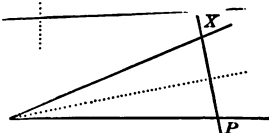
249. From a point P , without a given line AB , draw a line PX such that PXA shall equal a given angle C .



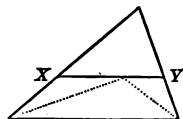
250. Find the bisector of the angle that would be formed by two given lines, without producing the lines.



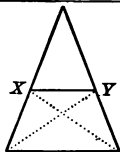
251. Through a given point P , draw a line that cuts off equal parts from two intersecting lines.



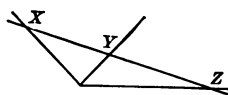
252. Draw an intercept parallel to the base of a triangle such that it shall be equal to the sum of the intercepts between it and the base.



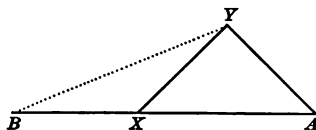
253. From a given isosceles triangle cut off a trapezoid having for base that of the triangle, and having its other three sides equal.



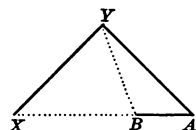
254. Three lines being given diverging from a point, draw a fourth line cutting them so that the intercepted segments shall be equal.



255. Construct an isosceles right triangle, the sum of the hypotenuse and a side being given.

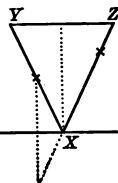


256. Construct an isosceles right triangle, the difference of the hypotenuse and a side being given.



257. Two angles of a triangle being given, find the third angle.

258. Construct an isosceles triangle of given altitude, whose sides pass through two given points, and whose vertex is in a given straight line.



Construct an isosceles triangle, having given :

259. The base and the vertical angle.

260. The base and a base angle.

261. An arm and the vertical angle.

Construct a triangle, having given :

262. Two sides and the included angle.

263. The base and the base angles.

264. The three sides, AB , AC , BC , such that $AC = \frac{1}{2} AB$, and $BC = \frac{1}{2} AC$.

265. Construct an isosceles right triangle, having given the sum and the difference of the hypotenuse and an arm.

HINT.—It is useful to remember that A and B , being any two magnitudes,

$$(A + B) + (A - B) = 2A; (A + B) - (A - B) = 2B.$$

266. Construct a right triangle, having given an arm and the altitude from the right angle upon the hypotenuse.

267. Construct a right triangle, having given the hypotenuse and the difference of the other sides.

Construct a parallelogram, having given :

268. Two adjacent sides and a diagonal.

269. A side and both diagonals.

270. Both diagonals and their included angle.

Construct a trapezoid, having given :

271. The four sides.

272. The parallel sides and the diagonals.

273. The parallel sides, a diagonal, and the angle formed by the diagonals.

274. Through a point within a circle, draw a chord that is bisected in that point, and show it is the least chord through that point.

275. The position and magnitude of two chords of a circle being given, describe the circle.

276. In a given circle, draw a chord whose length is double its distance from the center.

277. Draw that diameter of a given circle, which, being produced, meets a given line at a given distance from the center. When is this impossible ?

278. Describe a circle with given radius, to touch a given line in a given point. How many such circles can be described ?

279. Describe a circle of given radius to touch two intersecting lines. How many such circles can be described ?

280. Describe a circle touching two intersecting lines at a given distance from their intersection. How many such circles can be described ?

281. Describe a circumference passing through a given point, and touching a given line in a given point.

282. Describe a circumference touching two given lines, and passing through two given points between those lines.

283. From a given center, describe a circumference that bisects a given circumference.

284. With a given radius, describe a circle touching two given circles.

BOOK III.

RATIO. PROPORTION. LIMITS.



MEASUREMENT.

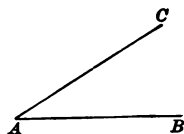
For many purposes, as in the propositions thus far considered, it is sufficient to prove, in regard to two given magnitudes, that they are equal or unequal. Thus we proved, in Prop. XXIV. (99), that $PA = PB$, and $PC > PA$. We have now to consider how to proceed when we wish to estimate exactly the relative greatness of given magnitudes.

220. To *measure* a magnitude is to find out how many times it contains another magnitude of the same kind.

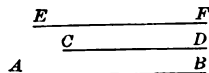
Thus we measure a line by finding how many times it contains another line called the *unit of length*, or *linear unit*. This unit may be either a standard unit, as an inch, a meter, etc., or a unit found by dividing a line into any desired number of equal parts, as in Prop. XIX. (207).

221. A *quantity* is a magnitude conceived as consisting of some number of equal parts.

Thus AB , regarded merely as a line, is a magnitude; but when thought of, or referred to, as 22 millimeters, or x linear units of any kind, it is a quantity measured by millimeters, or some other unit. The angle BAC , again, if referred to as an angle of $31^\circ 15' 47.2''$, is a quantity measured by tenths of seconds.



222. The number that expresses how many times a quantity contains the unit is called the *numerical measure* of that quantity. The numerical measure may be a number of any kind, integral or fractional, rational or irrational, or any letter denoting number.



Thus if AB is divided into 11 equal parts, of which CD is found to contain $8\frac{2}{3}$, and EF , x parts, then 11, $8\frac{2}{3}$, and x are the numerical measures of AB , CD , and EF , respectively.

If two quantities of the same kind, but expressed in different units, are to be compared as quantities, it is evident that both must be expressed as consisting of units of the same kind before the comparison can be effected.

Thus if AB were 8 rods and CD 15 yards, we cannot compare them until we have expressed them as 132 feet and 45 feet, for example. In every case, then, where numerical measures are compared, they are to be understood as referring to the same unit. It may also be observed that though abstract numbers are properly only the numerical measures of quantities, yet they are conveniently regarded as quantities whose unit is not expressed.

223. If a quantity is contained an exact number of times in each of two like quantities, it is called a *common measure* of these quantities, which are then said to be *commensurable* with each other.

Thus a rod and a yard are commensurable, since they have 2, 3, 6, 9 inches as common measures. $7\frac{1}{2}$ inches and $13\frac{1}{2}$ inches are commensurable, since they have $\frac{5}{8}$ of an inch as a common measure; and any two quantities are commensurable when, referred to the same unit, their numerical measures are rational numbers.

224. If two quantities have no common measure, they are said to be *incommensurable* with each other.

Thus every unit of our common system of measures is

incommensurable with every like unit of the metric system; and, as afterwards will be seen, the diagonal and side of a square are incommensurable with each other, as are also the diameter and circumference of a circle, etc. When such quantities are compared, their numerical measures are either approximate, as when we say 3 meters = 118.11237 + inches, or the numerical measures are such symbols as $\sqrt{2}$, $\sqrt{3}$, π , etc.

RATIO.

225. The *ratio* of two quantities is their relative greatness as expressed by the quotient of the one by the other.

Thus the ratio of 15 inches to 7 inches is $15 \text{ in.} \div 7 \text{ in.} = \frac{15}{7}$; the ratio of 8 rods to 15 yards is (reducing to a common measure) $132 \text{ ft.} \div 45 \text{ ft.} = \frac{132}{45}$; and, generally, if the numerical measures of two quantities are a and b , the ratio of these quantities is $\frac{a}{b}$, a and b being any numbers whatever.

226. The ratio of the quantity A to a like quantity B is denoted symbolically by either of the expressions

$$A : B, \text{ or } \frac{A}{B},$$

each of which is read, *the ratio of A to B*. In each, A is called the *first term* or *antecedent*; B , the *second term* or *consequent*.

227. When A and B are commensurable quantities, the value of their ratio is expressed exactly by the fraction denoting the quotient of the numerical measure of the antecedent by that of the consequent. But if the quantities are incommensurable, no rational fraction can express their ratio exactly, since then they would not be incommensurable. Yet, by taking the unit of measure sufficiently small, we can find a fraction that expresses the true value of the ratio to as near a degree of approximation as we please.

Thus suppose we have two lines A and B , whose numerical measures are $\sqrt{2}$ and 1, respectively. Now, carried to seven decimals, $\sqrt{2} = 1.4142135 +$; that is, $\sqrt{2} > 1.414213$ and < 1.414214 , so that the ratio of A to B or $\sqrt{2}:1$ lies between

$$\frac{1414213}{1000000} \text{ and } \frac{1414214}{1000000},$$

and must differ from either by less than one millionth. As we can find the value of $\sqrt{2}$ to any number of decimals, we can find a fraction that differs from $\sqrt{2}:1$ by less than any assignable quantity.

To generalize, let A and B be any two incommensurable quantities. If we suppose B divided into any number of equal parts, so that $B = nP$, P denoting one of the parts, then A must contain some number m of such parts, with a remainder less than P ; or $A > mP$ and $< (m+1)P$. Thus

$$\frac{A}{B} > \frac{mP}{nP} \text{ and } < \frac{(m+1)P}{nP};$$

that is, $\frac{A}{B}$ lies between $\frac{m}{n}$ and $\frac{m+1}{n}$, and must differ from either of these fractions by less than $\frac{1}{n}$, a fraction that may be made less than any assigned quantity by taking n sufficiently great. Hence

A rational fraction can be found that expresses the ratio of any two given incommensurable quantities within any required degree of precision.

PROPORTION. DEFINITIONS.

228. If two pairs of quantities have equal ratios, they are said to be *proportionals* or to be *in proportion*. Thus each of the equalities $A:B = C:D$, $\frac{A}{B} = \frac{C}{D}$,

expresses a proportion that may be read, *the ratio of A to B is equal to the ratio of C to D*; or more briefly, *A is to B as C is to D*.

229. The first and fourth terms of a proportion are called its *extremes*; the second and third, its *means*. The fourth term is also called a *fourth proportional* to the other three.

230. Although ratio, from its very nature, can exist between like quantities only, yet as we may have, for example, $\angle A = \frac{5}{8} \angle B$, and also $\text{arc } P = \frac{5}{8} \text{ arc } Q$, the proportion

$$\angle A : \angle B = \text{arc } P : \text{arc } Q,$$

may be stated between these pairs of unlike quantities, since the proportion simply states that the angle *A* is just as great compared with the angle *B* as is the arc *P* compared with the arc *Q*.

231. Of the following theorems concerning proportions and their transformations, some apply to pairs of quantities whether like or unlike, and the given proportion will be stated under the form

$$A : B = P : Q.$$

Others apply only when the pairs of quantities are like, and the given proportions will be stated under the form

$$A : B = C : D.$$

Others, again, that apply properly to numbers only, will have the given proportion stated under the form

$$a : b = c : d.$$

232. It follows at once, from the definitions of *quantity*, *ratio*, and *proportion*, that (1) *if four quantities are in proportion, their numerical measures are also in proportion*; i.e.

if

$$A : B = P : Q,$$

then

$$a : b = p : q, \text{ or } \frac{a}{b} = \frac{p}{q},$$

and conversely; (2) *if four numbers are in proportion, quantities of which these numbers are numerical measures are also in proportion; i.e.,*

if $a : b = p : q$, or $\frac{a}{b} = \frac{p}{q}$,

then $A : B = P : Q$.

It also follows from the same definitions, and Ax. 1, that
(3) *ratios equal to the same ratio are equal to each other.*

EXERCISES.

QUESTIONS.

285. Taking the inch as unit, what is the ratio of 1 ft. to 7 in.? To 13 in.? To $1\frac{1}{2}$ ft.? To $2\frac{1}{2}$ ft.? To $\frac{1}{2}$ yd.?

286. A train goes at the rate of 112 miles in $3\frac{1}{2}$ hrs.; a second train at the rate of 105 miles in $2\frac{1}{2}$ hrs. What is the ratio of the speed of the first train to that of the second?

287. What is the ratio of 1 lb. to 9 oz.? To 33 oz.? To $2\frac{1}{2}$ lbs.? To $20\frac{1}{2}$ lbs.?

288. That 216 grs. of silver may be worth $13\frac{1}{2}$ grs. of gold, what should be the ratio of the value of gold to that of silver?

289. 180° F. = 100° C. = 80° R. What is the ratio of 1° F. to 1° C. and 1° R., respectively?

290. The vertical angle of an isosceles triangle is 50° . What is the ratio of that angle, (1) to a right angle? (2) to each of the base angles?

291. A base angle of an isosceles triangle is 75° . What is the ratio of that angle, (1) to a right angle? (2) to the vertical angle?

292. The ratio of a base angle of an isosceles triangle to the vertical angle is $\frac{5}{8}$. What is that angle, and what is its ratio to a right angle?

293. An acute angle of a right triangle is 35° . What is the ratio of that angle to the other acute angle?

294. A certain angle has to an angle of an equilateral triangle the same ratio that the latter has to a right angle. How many degrees in the first angle?

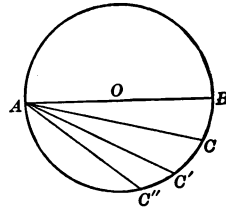
LIMITS.

Thus far the magnitudes considered have been of fixed greatness only. In Art. 227 we had occasion, however, to discuss certain approximate ratios tending towards a value they can never exactly attain, though they can approach it indefinitely near. In these we have examples of what are termed a *variable* and its *limit*, now about to be defined.

233. A *constant* is a magnitude or quantity whose greatness remains the same, neither increasing nor decreasing.

234. A *variable* is a magnitude or quantity whose greatness goes on increasing or decreasing.

Thus a chord AB , as long as it passes through the center O , remains constant in every position. But if it turn about an extremity A , we have a *decreasing* variable chord AC , AC' , etc., while in the angles BAC , etc., and in the arcs BC , BC' , etc., we have *increasing* variables.



235. If a variable increases or decreases so as to approach indefinitely near to an equality with a certain constant, this constant is called the *limit* of the variable.

Thus in the figure above, the variable chord decreases towards zero as limit, the variable angle increases towards a right angle as limit, the variable arc increases towards a semicircumference as limit, and the distance of the variable chord from the center increases towards the radius as limit, none of which limits, evidently, can be attained as long as there is any chord.

Again, if a point P $O \quad P \quad P' \quad P'' \quad P''' \quad N$
move along from O
towards N , the distances OP and PN are variables; the one increasing towards ON , the other decreasing towards zero. If no condition were imposed upon the motion of P , it might

reach N , or even pass beyond it, and the distance ON would not be a limit of OP according to the definition. But if we impose the condition that at the end of the first second it reach P' , half way to

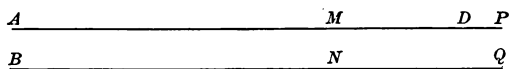
N , at the end of the $O \quad P \quad P' \quad P'' \quad P''' \quad N$
next second reach P''

half way between P' and N , and so on, it is evident that P could never reach N , though it might come indefinitely near to it. For the fraction of the distance passed over in n seconds would be the sum of the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$ a series that has 1 for its limit, a limit the series can never attain, no matter how many terms may be taken.



PROPOSITION I. THEOREM.

236. *If two variables tending towards limits are always equal, these limits are also equal.*



Given: Two equal variables AM and BN , tending towards limits AP and BQ ;

To Prove: BQ is equal to AP .

For if AP could be greater than BQ , some part of AP , as AD , would be equal to BQ . Now, however small the constant difference DP could be, since AM can increase so as to approach AP nearer than any constant difference (Hyp.), AM would become greater than AD or its equal BQ , while BN would always remain less than BQ (Hyp.). That is, BN would be both equal to and less than AM . In the same way it can be proved that BQ cannot be greater than AP .

Hence BQ must be equal to AP . Q.E.D.

PROPORTION. THEOREMS.

PROPOSITION II. THEOREM.

1. **237.** *If four numbers are in proportion, the product of the extremes is equal to the product of the means.*

Given : $a : b = c : d ;$

To Prove : $ad = bc.$

Since $\frac{a}{b} = \frac{c}{d},$ (Hyp.)

\therefore multiplying both members by $bd,$

$ad = bc.$ Q.E.D.

238. COR. *If the means are equal, that is, if*

$$a : b = b : c,$$

then $ac = b^2,$ (237)

i.e., the product of the extremes is equal to the square of the mean.

239. DEFINITION. Here b is said to be a *mean proportional* between a and c , and c is a *third proportional* to a and b .

EXERCISE 295. If the bisector of an angle of a parallelogram passes through the opposite vertex, the figure must be equilateral.

296. Any parallelogram that can be circumscribed about a circle must be equilateral.

297. If each of three equal circles is tangent to the other two, their lines of centers form an equilateral triangle.

298. In triangle ABC , if D, E , the mid points of AB, AC , be joined, show that $\triangle ABC : \text{trapezoid } DBCE = 4 : 3$.

299. If A is an angle of an equilateral triangle and B is a right angle, show that $\angle A : \angle B = 2 : 3$.

300. In the diagram for Art. 153, if $\angle BAC : \angle ACB = 7 : 8$, and if $\angle BAC : \angle ABC = 7 : 3$, show that $\angle AOC : \angle ABC = 7 : 2$.

301. Hence deduce the ratio of angle BOC to angle BAC and of angle AOB to angle ACB .

PROPOSITION III. THEOREM.

2. 240. *Conversely, if the product of two numbers is equal to the product of two others, either two may be made the extremes of a proportion, and the other two its means.*

Given: $ad = bc$;
 To Prove: $a : b = c : d$.

Since $ad = bc$, (Hyp.)
 \therefore dividing both members by bd ,

$$\frac{a}{b} = \frac{c}{d},$$

i.e., $a : b = c : d$. Q.E.D.

241. SCHOLIUM. If, instead of by bd , we divide by ab , ac , or cd , we obtain from $ad = bc$ the proportions, each derivable from any of the others:

$$d : b = c : a, \text{ or } c : a = d : b;$$

$$d : c = b : a, \text{ or } b : a = d : c;$$

$$a : c = b : d, \text{ or } b : d = a : c.$$

EXERCISE 302. In the first diagram for Prop. XIII. (195), prove that if AC and BD be joined, angle BAC will be equal to angle ABD .

303. In the same diagram, show that if AD and BC be joined, they will intersect in MN .

304. In the third diagram for the same proposition, if a perpendicular be drawn through the mid point of EF , it will pass through M and N .

305. If two tangents to the same circle make equal angles with an intercept between them that passes through the center, they are equal.

306. If two secants to the same circle make equal angles with an intercept between them that passes through the center, they are equal.

PROPOSITION IV. THEOREM.

10. 242. The products of the corresponding terms of two or more numerical proportions are in proportion.

Given : $a : b = c : d$, and $e : f = g : h$;

To Prove : $ae : bf = cg : dh$.

Since $\frac{a}{b} = \frac{c}{d}$, and $\frac{e}{f} = \frac{g}{h}$, (Hyp.)

\therefore multiplying member by member,

$$\frac{ae}{bf} = \frac{cg}{dh},$$

i.e., $ae : bf = cg : dh$. Q.E.D.

In the same way the theorem can be proved for any number of proportions.



PROPOSITION V. THEOREM.

11. 243. If four numbers are in proportion, like powers or like roots of those numbers are in proportion.

Given : $a : b = c : d$;

To Prove : $a^n : b^n = c^n : d^n$, and $a^{\frac{1}{n}} : b^{\frac{1}{n}} = c^{\frac{1}{n}} : d^{\frac{1}{n}}$.

Since $\frac{a}{b} = \frac{c}{d}$, (Hyp.)

\therefore taking like powers or like roots of both members,

$$\frac{a^n}{b^n} = \frac{c^n}{d^n} \text{ and } \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \frac{c^{\frac{1}{n}}}{d^{\frac{1}{n}}},$$

i.e., $a^n : b^n = c^n : d^n$, and $a^{\frac{1}{n}} : b^{\frac{1}{n}} = c^{\frac{1}{n}} : d^{\frac{1}{n}}$ Q.E.D.

PROPOSITION VI. THEOREM.

¶ 244. *If four like quantities are in proportion, they are in proportion taken alternately.*

Given: $A : B = C : D ;$

To Prove: $A : C = B : D.$

Since $\frac{a}{b} = \frac{c}{d}$ (Hyp. and 232')

$$ad = bc, \quad (237)$$

$$\therefore a : c = b : d, \quad (241)$$

$$\therefore A : C = B : D. \quad \text{Q.E.D. (232'')}$$

SCHOLIUM. *Alternation*, as will be seen from the symbolical statement of the theorem, means taking the alternate terms of a given proportion so as to form a new one.

It is also evident that this transformation can be applied only to a proportion in which the terms are all like terms, since, if A and P are unlike quantities, no ratio can exist between them.

 PROPOSITION VII. THEOREM.

¶ 245. *If any four quantities are in proportion, they are in proportion taken inversely.*

Given: $A : B = P : Q ;$

To Prove: $B : A = Q : P.$

Since $a : b = p : q,$ (Hyp. and 232')

$$bp = aq, \quad (237)$$

$$\therefore b : a = q : p, \quad (241)$$

$$\therefore B : A = Q : P. \quad \text{Q.E.D. (232'')}$$

SCHOLIUM. *Inversion*, as may be seen from the symbolical statement, means taking the terms in inverse order.

PROPOSITION VIII. THEOREM.

6. 246. *If any four quantities are in proportion, they are in proportion taken by composition.*

Given: $A : B = P : Q;$

To Prove: $A + B : B = P + Q : Q.$

Since $\frac{a}{b} = \frac{p}{q},$ (Hyp. and 232')

$\frac{a}{b} + 1 = \frac{p}{q} + 1,$ (Ax. 2)

$\therefore \frac{a+b}{b} = \frac{p+q}{q},$

$\therefore A + B : B = P + Q : Q.$ Q.E.D. (232'')

Similarly, $A + B : A = P + Q : P.$



PROPOSITION IX. THEOREM.

6. 247. *If any four quantities are in proportion, they are in proportion taken by division.*

Given: $A : B = P : Q;$

To Prove: $A - B : B = P - Q : Q.$

Since $\frac{a}{b} = \frac{p}{q},$ (Hyp. and 232')

$\frac{a}{b} - 1 = \frac{p}{q} - 1,$ (Ax. 3)

$\therefore \frac{a-b}{b} = \frac{p-q}{q},$

$\therefore A - B : B = P - Q : Q.$ Q.E.D. (232'')

Similarly, $A - B : A = P - Q : P.$

PROPOSITION X. THEOREM.

7, 243. *If any four quantities are in proportion, they are in proportion taken by composition and division.*

Given: $A : B = P : Q;$

To Prove: $A + B : A - B = P + Q : P - Q.$

Since $a : b = p : q,$ (Hyp. and 232')

$$\frac{a+b}{b} = \frac{p+q}{q}, \quad (246)$$

$$\text{and } \frac{a-b}{b} = \frac{p-q}{q}, \quad (247)$$

\therefore dividing member by member,

$$\frac{a+b}{a-b} = \frac{p+q}{p-q},$$

$$\therefore A + B : A - B = P + Q : P - Q. \quad \text{Q.E.D.} \quad (232'')$$

PROPOSITION XI. THEOREM.

249. *If two proportions have the same antecedents, the consequents are in proportion, and vice versa.*

Given: $A : B = P : Q,$ and $A : C = P : R;$

To Prove: $B : C = Q : R.$

$$\left. \begin{array}{l} \text{Since } \frac{a}{c} = \frac{p}{r} \\ \text{and } \frac{a}{b} = \frac{p}{q} \end{array} \right\} \quad (\text{Hyp. and 232'})$$

\therefore dividing member by member,

$$\frac{b}{c} = \frac{q}{r},$$

$$\therefore B : C = Q : R. \quad \text{Q.E.D.} \quad (232'')$$

250. DEFINITION. *A continued proportion is a series of equal ratios.*

PROPOSITION XII. THEOREM.

¶ 251. *If any number of like quantities are in continued proportion, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Given: $A : B = C : D = E : F ;$

To Prove: $A + C + E : B + D + F = A : B.$

Since $a : b = c : d = e : f,$ (Hyp. and 232')

$$ab = ba, ad = bc, af = be, \quad (237)$$

$$\therefore a(b + d + f) = b(a + c + e),$$

$$\therefore a + c + e : b + d + f = a : b, \quad (240)$$

$$\therefore A + C + E : B + D + F = A : B. \quad \text{Q.E.D.} \quad (232'')$$

In the same way, the theorem may be proved for any number of ratios.



PROPOSITION XIII. THEOREM.

¶ 252. *If the antecedents of any four quantities in proportion be multiplied by any number, and the consequents by any, the results will be in proportion.*

Given: $A : B = P : Q ;$

To Prove: $mA : nB = mP : nQ.$

Since $a : b = p : q,$ (Hyp. and 232')

and

$$m : n = m : n,$$

$$ma : nb = mp : nq, \quad (242)$$

$$\therefore mA : nB = mP : nQ. \quad \text{Q.E.D.} \quad (232'')$$

PROPOSITION XIV. THEOREM.

253. *If both terms of a ratio be multiplied or divided by any number, the ratio remains the same.*

Given : **Any ratio** $A : B$;

To Prove : $mA : mB = A : B = \frac{1}{n}A : \frac{1}{n}B$.

Since $a : b = a : b = a : b$,

and $m : m = 1 : 1 = \frac{1}{n} : \frac{1}{n}$,

$$ma : mb = a : b = \frac{1}{n}a : \frac{1}{n}b, \quad (242)$$

$$\therefore mA : mB = A : B = \frac{1}{n}A : \frac{1}{n}B. \quad \text{Q.E.D. (232'')} \quad (243)$$

254. COR. *If* $A : B = P : Q$,

then

$$mA : mB = nP : nQ,$$

and

$$\frac{1}{m}A : \frac{1}{m}B = \frac{1}{n}P : \frac{1}{n}Q. \quad (253)$$

EXERCISE 307. Any parallelogram that can be inscribed in a circle will have the intersection of its diagonals at the center of the circle.

308. Hence, show that rectangles are the only parallelograms that can be inscribed in a circle.

309. Any parallelogram that can be inscribed in a circle must be equiangular.

310. Describe a circumference passing through two given points and having its center in a given straight line. When is this impossible?

311. Prove that all circumferences that pass through a given point and have their centers in a given straight line must also pass through a second given point.

312. From a given point as center describe a circle to touch a given circle. How many solutions are there?

PROPOSITION XV. THEOREM.

255. *Two incommensurable ratios are equal if their corresponding approximate values are always equal.*

Given: Two ratios $\frac{A}{B}$ and $\frac{P}{Q}$ such that, when an approximate value of $\frac{A}{B}$ is $\frac{m}{n}$, the corresponding value of $\frac{P}{Q}$ also is $\frac{m}{n}$;

To Prove: $A : B = P : Q$.

The supposition is that when B and Q are each divided into n equal parts, if A contain m parts of B , with some remainder, then also P contains m parts of Q , with some remainder.

$$\text{Since } \frac{A}{B} = \frac{m}{n} + \frac{x}{n} \text{ and } \frac{P}{Q} = \frac{m}{n} + \frac{x'}{n}, \quad (\text{Hyp.})$$

m and n being integers, but x and x' each < 1 ,

$$\left. \begin{array}{l} \therefore \frac{m}{n} \text{ has for limit } \frac{A}{B}, \\ \text{also } \frac{m}{n} \text{ has for limit } \frac{P}{Q}. \end{array} \right\} \quad (235)$$

as $\frac{x}{n}, \frac{x'}{n}$ each tend toward zero when n is indefinitely great,

$$\therefore \frac{A}{B} = \frac{P}{Q} \quad (236)$$

being the limits of variables always equal,

$$\text{i.e., } A : B = P : Q.$$

Q.E.D.

SCHOLIUM 1. It is to be carefully noted that $A : B$ and $P : Q$ are here proved *absolutely*, not *approximately*, equal.

SCHOLIUM 2. In Props. II.–XIV., we found that, a proportion being given, certain transformations can be performed on it. In this proposition we have the criterion, or test, of proportionality as regards both incommensurable quantities and those of whose commensurability we know nothing.

BOOK IV.

PROPORTIONAL ANGLES AND LINES. SIMILAR POLYGONS.

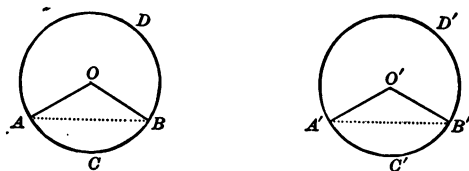


PROPORTIONAL ANGLES.

256. DEFINITION. An angle formed by two radii of a circle is called an *angle at the center*, and is said to *intercept* the arc that lies between its sides.

PROPOSITION I. THEOREM.

257. *In the same circle, or in equal circles, equal angles at the center intercept equal arcs; and conversely.*



1°. *Given:* At the centers of equal circles ADB , $A'D'B'$, angle O equal to angle O' ;

To Prove: Arc ACB is equal to arc $A'C'B'$.

Join AB , $A'B'$. Then

since $OA = O'A'$, $OB = O'B'$, and $\angle O = \angle O'$, (Hyp.)

$$\triangle OAB = \triangle O'A'B', \quad (66)$$

$$\therefore AB = A'B', \quad (70)$$

$$\therefore \text{arc } ACB = \text{arc } A'C'B'. \quad \text{Q.E.D. } (174'')$$

2°. *Given:* In equal circles ADB , $A'D'B'$, equal arcs ACB , $A'C'B'$, intercepted by angles O , O' , respectively;

To Prove: Angle O is equal to angle O' .

Join AB and $A'B'$. Then
 since arc $ACB =$ arc $A'C'B'$, (Hyp.)
 chord $AB =$ chord $A'B'$; (174")
 also $OA = O'A'$, and $OB = O'B'$, (Hyp.)
 $\therefore \angle O = \angle O'$. Q.E.D. (69, 70)

258. COR. 1. *A radius bisecting an angle at the center bisects its arc; and conversely.*

259. COR. 2. *If angle A is equal to m times angle B , then the arc of angle A is equal to m times the arc of angle B ; and conversely.*

SCHOLIUM. The conclusion proved in Prop. I. might be stated under the form

$$\angle O : \angle O' = m : m = \text{arc } ACB : A'C'B',$$

such a ratio as $m : m$ being called a *ratio of equality*.

EXERCISE 313. In the same circle or in equal circles, the greater of two unequal angles at the center intercepts the greater arc.

314. An angle at the center is obtuse, right, or acute, according as its arc is greater than, equal to, or less than a quarter of a circumference.

315. Intersecting diameters intercept equal arcs between their extremities.

316. If the extremities of intersecting diameters be joined, the figure formed will be a rectangle.

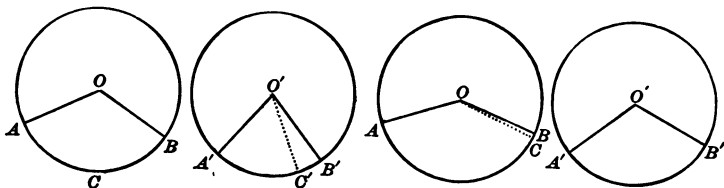
317. In the diagram for Prop. I., if angle O is $\frac{1}{2}$ of a right angle, what is the ratio of arc ACB to arc ADB ?

318. In the same diagram, if AO be produced to meet the circumference in E , what part will arc BE be of the circumference, supposing, as before, that angle O is $\frac{1}{2}$ of a right angle?

319. In the left-hand circle of the diagram for Prop. II., show how to bisect arc ACB without joining AB .

PROPOSITION II. THEOREM.

260. *In the same circle, or in equal circles, angles at the center are proportional to their intercepted arcs.*



Given: In equal circles with centers O and O' , respectively, angles AOB , $A'O'B'$, with their respective arcs AB , $A'B'$;

To Prove: Angle AOB : angle $A'O'B'$ = arc AB : arc $A'B'$.

1°. When the arcs AB , $A'B'$, are commensurable.

Let $B'C'$ be a common measure of $A'B'$ and AB , so that arc $B'C'$ is contained 5 times in $A'B'$ and 8 times in AB . Join $O'C'$.

Since arc $AB = 8$ arc $B'C'$, and arc $A'B' = 5$ arc $B'C'$, (Hyp.)

$$\angle AOB = 8 \angle B'O'C', \text{ and } \angle A'O'B' = 5 \angle B'O'C', \quad (259)$$

$$\therefore \left. \begin{aligned} \text{arc } AB : \text{arc } A'B' &= 8 : 5, \\ \text{and } \angle AOB : \angle A'O'B' &= 8 : 5, \end{aligned} \right\} \quad (252)$$

$$\therefore \angle AOB : \angle A'O'B' = \text{arc } AB : \text{arc } A'B'. \quad \text{Q.E.D.} \quad (232''')$$

2°. When the arcs AB , $A'B'$, are incommensurable.

Suppose $A'B'$ divided into any number of equal parts n , and that AB contains this n th part of $A'B'$ m times, with a remainder BC . Draw OC .

Since AC and $A'B'$ are commensurable arcs, (Const.)

$$\frac{\angle AOC}{\angle A'O'B'} = \frac{m}{n} = \frac{\text{arc } AC}{\text{arc } A'B'} \quad (1^\circ)$$

$$\therefore \frac{\angle AOB}{\angle A'O'B'} = \frac{m}{n} + \frac{x}{n}, \text{ and } \frac{\text{arc } AB}{\text{arc } A'B'} = \frac{m}{n} + \frac{x'}{n},$$

since AOB and AB are slightly $> AOC$ and AC , respectively.

Now when n is taken indefinitely great, $\frac{x}{n}$ and $\frac{x'}{n}$ become indefinitely small,

x and x' being each < 1 ;

$$\therefore \frac{\angle AOB}{\angle A'O'B'} = \frac{\text{arc } AB}{\text{arc } A'B'} \quad \text{Q.E.D.} \quad (255)$$

being the limits of variables always equal.

261. DEFINITION. One quantity is said to be *measured* by another of a different kind, if they are so related that the numerical measure of the one always expresses that of the other also.

This may also be worded as follows: A quantity A is measured by another quantity A' , if A and its unit U are always in proportion to A' and its unit U' ;

i.e., if $A : U = A' : U'$.

Thus *temperature* is measured by the number of equal *lengths* in a column of mercury; *time*, by the number of equal *arcs* in the circumference of a dial; etc.

262. COR. *An angle at the center is measured by its intercepted arc.*

For if A denote any angle, and U its unit-angle, A' the intercepted arc, and U' its unit-arc, then (260),

$$\angle A : \angle U = \text{arc } A' : \text{arc } U' = m : 1,$$

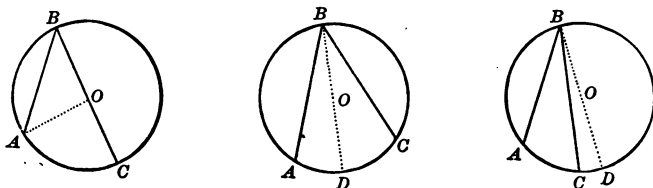
m being the numerical measure of A' in terms of its unit U' .

SCHOLIUM. Though any convenient arc may be taken as unit, that usually employed is the 90th part of a *quadrant* or $\frac{1}{4}$ of a circumference, and is called a *degree*. As a right \angle is measured by a quadrant (262), the 90th part of a right \angle is also called a degree. Each degree, again, is divided into 60 *minutes*, and each minute into 60 *seconds*.

263. DEFINITION. The angle formed by two chords that meet in the circumference is called an *inscribed angle*.

PROPOSITION III. THEOREM.

264. *An inscribed angle is measured by half its intercepted arc.*



Given : An inscribed angle ABC intercepting the arc AC ;

To Prove : Angle ABC is measured by $\frac{1}{2}$ arc AC .

Find O , the center of the circle. Then

1°. If O lies in a side BC of $\angle ABC$, join AO .

$$\text{Since } AO = BO, \quad (162)$$

$$\angle A = \angle B. \quad (68)$$

$$\text{But } \angle AOC = \angle A + \angle B = 2\angle B, \quad (122)$$

$$\therefore \angle B = \frac{1}{2} \angle AOC. \quad (\text{Ax. 7})$$

$$\text{Now } \angle AOC \text{ is meas. by arc } AC, \quad (262)$$

$$\angle B \text{ is meas. by } \frac{1}{2} \text{ arc } AC. \quad \text{Q.E.D.}$$

2°. If O lies between BA and BC , draw BOD , a diam.

$$\left. \begin{array}{l} \text{Since } \angle ABD \text{ is meas. by } \frac{1}{2} \text{ arc } AD, \\ \text{and } \angle DBC \text{ is meas. by } \frac{1}{2} \text{ arc } DC, \end{array} \right\} \quad (1^\circ)$$

$$\angle ABD + \angle DBC \text{ is meas. by } \frac{1}{2} (\text{arc } AD + \text{arc } DC);$$

$$\text{i.e., } \angle ABC \text{ is meas. by } \frac{1}{2} \text{ arc } AC. \quad \text{Q.E.D.}$$

3°. If O lies without BA and BC , draw BOD , a diam.

$$\left. \begin{array}{l} \text{Since } \angle ABD \text{ is meas. by } \frac{1}{2} \text{ arc } AD, \\ \text{and } \angle DBC \text{ is meas. by } \frac{1}{2} \text{ arc } DC, \end{array} \right\} \quad (1^\circ)$$

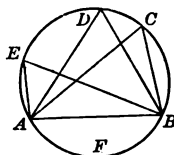
$$\angle ABD - \angle DBC \text{ is meas. by } \frac{1}{2} (\text{arc } AD - \text{arc } DC);$$

$$\text{i.e., } \angle ABC \text{ is meas. by } \frac{1}{2} \text{ arc } AC. \quad \text{Q.E.D.}$$

265. DEFINITION. A *segment* of a circle is the figure contained by a chord and its arc.

266. COR. 1. All angles C, D, E , inscribed in the segment $AEDCB$ of a circle are equal.

For each is measured by half the arc AFB .



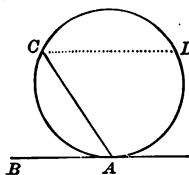
267. COR. 2. An angle inscribed in a semicircle is a right angle.

For it is measured by $\frac{1}{2}$ a semicircumference; i.e., by $\frac{1}{4}$ a circumference, or a quadrant.

268. COR. 3. The arc intercepted by an inscribed angle is double the arc intercepted by an equal angle at the center.

PROPOSITION IV. THEOREM.

269. The angle formed by a tangent and a chord meeting at the point of contact, is measured by half the intercepted arc.



Given: An angle BAC formed by a tangent AB and a chord AC ;

To Prove: Angle BAC is measured by $\frac{1}{2}$ arc AC .

Through C draw $CD \parallel$ to AB (109). Then

since CD is \parallel to AB , (Const.)

arc $AD =$ arc AC , (195")

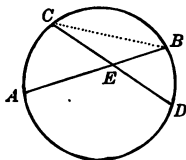
and $\angle C = \angle A$. (110")

But $\angle C$ is meas. by $\frac{1}{2}$ arc AD , (264)

$\therefore \angle A$ is meas. by $\frac{1}{2}$ arc AD or $\frac{1}{2}$ arc AC . Q.E.D.

PROPOSITION V. THEOREM.

270. *The vertical angles formed by intersecting chords are each measured by half the sum of the intercepted arcs.*



Given: Two chords AB , CD , intersecting in E ;

To Prove: Angles AEC , BED , are each measured by $\frac{1}{2}$ (arc AC + arc BD).

Draw the chord BC . Then

since $\angle B$ is meas. by $\frac{1}{2}$ arc AC ,

and $\angle C$ is meas. by $\frac{1}{2}$ arc BD , (264)

also $\angle AEC = \angle B + \angle C$, (122)

$\angle AEC$ or $\angle BED$ is meas. by $\frac{1}{2}$ (arc AC + arc BD). Q.E.D.

EXERCISE 320. In the diagram for Prop. III., if B is an angle of 32° , how many degrees are there in arc AC ?

321. In the diagram for Prop. IV., if CD is an arc of 102° , then BAC is an angle of how many degrees?

322. In the diagram for Prop. V., if AEC is an angle of 25° and AC an arc of 30° , how many degrees in arc BD ?

323. Any three points of a circumference being given, how can we find other points of it without knowing the center?

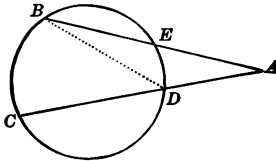
324. The opposite sides of an inscribed parallelogram divide in the same ratio the radii drawn perpendicular to them.

325. If two circles whose centers are O and O' have a common tangent AB , and OA , $O'B$, be joined, these lines will be parallel.

326. If two tangents, PA , PB , be drawn to a circle whose center is O , and AB , AO , be drawn, then will angle $BAO = \frac{1}{2}$ angle P .

PROPOSITION VI. THEOREM.

271. *The angle formed by two secants meeting without the circle is measured by half the difference of the intercepted arcs.*



Given: Secants AB, AC , meeting in A , and intercepting arcs BC, DE ;

To Prove: Angle A is measured by $\frac{1}{2}(\text{arc } BC - \text{arc } DE)$.

Draw the chord BD . Then

$$\text{since } \angle BDC = \angle A + \angle B, \quad (122)$$

$$\angle A = \angle BDC - \angle B. \quad (\text{Ax. } 3)$$

$$\begin{aligned} \text{But } \angle BDC \text{ is meas. by } \frac{1}{2} \text{ arc } BC, \\ \text{and } \angle B \text{ is meas. by } \frac{1}{2} \text{ arc } DE, \end{aligned} \quad (264)$$

$$\therefore \angle A \text{ is meas. by } \frac{1}{2}(\text{arc } BC - \text{arc } DE). \quad \text{Q.E.D.}$$

EXERCISE 327. In the diagram for Prop. VI., if A is an angle of 17° and DE an arc of 36° , how many degrees in arc BC ?

328. If a polygon of an even number of sides be inscribed in a circle, the sums of its alternate angles are equal.

329. Find a point equidistant from three given points. When is the problem impossible?

330. Under what conditions is it possible to find a point equidistant from four given points?

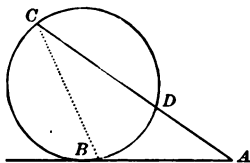
331. Prove the theorem given in Art. 155, by means of Prop. III., (171).

332. If an inscribed triangle has unequal angles, the greater angle intercepts the greater arc.

333. If two circles intersect, their line of centers produced will bisect each of the four arcs.

PROPOSITION VII. THEOREM.

272. *An angle formed by a tangent and a secant is measured by half the difference of the intercepted arcs.*



Given: Tangent AB and secant AC meeting in A and intercepting arcs BC , BD ;

To Prove: Angle A is measured by $\frac{1}{2}(\text{arc } BC - \text{arc } BD)$.

Draw the chord BC . Then

$$\text{since } \angle B = \angle A + \angle C, \quad (122)$$

$$\angle A = \angle B - \angle C. \quad (\text{Ax. } 3)$$

$$\text{But } \angle B \text{ is meas. by } \frac{1}{2} \text{ arc } BC, \quad (269)$$

$$\text{and } \angle C \text{ is meas. by } \frac{1}{2} \text{ arc } BD, \quad (264)$$

$$\therefore \angle A \text{ is meas. by } \frac{1}{2}(\text{arc } BC - \text{arc } BD). \quad \text{Q.E.D.}$$

273. COR. *The angle formed by two tangents is measured by half the difference of the intercepted arcs.*



EXERCISES.

QUESTIONS.

334. If an angle A is measured by two thirds of a quadrant, and an angle $B = 50^\circ$, what is the ratio of $\angle A$ to $\angle B$?

335. By what fraction of a quadrant is the vertical angle of an isosceles triangle measured, (1) if it is twice as great as a base angle?

(2) if it is $\frac{1}{2}$ as great? (3) if it is $\frac{1}{n}$ th as great?

336. Two angles of a triangle are measured by $\frac{1}{3}$ and $\frac{1}{4}$ of a circumference, respectively. How many degrees in each of the three angles?

337. One angle of a right triangle is measured by $\frac{1}{10}$ of a circumference. How many degrees in the other acute angle, and what is its ratio to the first?

338. If, in the diagram for Cor. I. of Prop. III., ADB is an arc of 230° , how many degrees are there in each of the angles C , D , and E ?

339. If, further, in the same diagram, AE is an arc of 30° , how many degrees in angles ABE and BAE , respectively?

340. In the diagram for Prop. IV., if BAC is an angle of 65° , how many degrees are there in arc CD ?

341. In the same diagram, if arc $CA = \frac{2}{3}$ arc CD , how many degrees in $\angle BAC$?

342. In the diagram for Prop. V., if AEC is an angle of 40° , how many degrees are there in the sum of the arcs AD and BC ?

343. In the diagram for Prop. VI., if A is an angle of 25° , and arc $DE = \frac{1}{2}$ arc BC , how many degrees are there in the sum of arcs BE and CD ?

344. In the same diagram, if $BD = AD$, how many degrees are there in each of the arcs BC and DE , if $\angle A = n$ degrees?

345. In the diagram for Prop. VII., if arc $BC = 3$ arc BD , and $\angle A = 30^\circ$, how many degrees are there in arc CD and in $\angle B$?

346. In the same diagram, if $BA = BC$, what is the ratio of arc BC to arc BD ?

347. If the angle formed by two tangents is 30° , how many degrees are there in each of the intercepted arcs?

348. If a tangent be drawn at a vertex of an inscribed equilateral triangle, what angle will it form with either adjacent side?

349. If from the same point in a circumference, a side of a square and a side of an equilateral triangle be inscribed, the difference of the arcs subtended by them will be what part of the circumference?

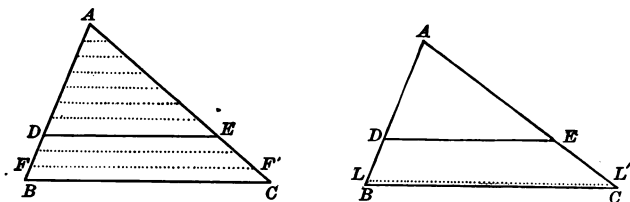
350. If the vertical angle of an inscribed isosceles triangle is 54° , what is the ratio of the arc opposite that angle to either of the other arcs?

351. If the vertical angle were $37^\circ 15' 32''$, what would those ratios be?

PROPORTIONAL LINES.

PROPOSITION VIII. THEOREM.

§ 274. *A line drawn parallel to one side of a triangle and meeting the other two sides, divides them proportionally.*



Given: DE parallel to BC , a side of triangle ABC , and meeting AB , AC , in D , E , respectively;

To Prove: $DB : AD = EC : AE$.

1°. When AD and DB are commensurable.

Let BF be a common measure of DB and AD , so that BF can be laid off 3 times on DB and 7 times on AD .

Through the points of division draw lines FF' , etc., \parallel to BC .

Since the \parallel 's FF' , etc., cut off 3 and 7 equal parts on DB , AD , respectively, (Const.)

the \parallel 's FF' , etc., cut off 3 and 7 equal parts on EC , AE , respectively; (151)

$$\therefore DB : AD = 3 : 7, \text{ and } EC : AE = 3 : 7, \quad (225)$$

$$\therefore DB : AD = EC : AE. \quad \text{Q.E.D.} \quad (232''')$$

2°. When AD and DB are incommensurable.

Suppose AD divided into any number of equal parts n , and that DB contains this n th part of AD m times, with a remainder LB . Through L draw $LL' \parallel$ to BC .

Since AD and DL are commensurables, (Const.)

$$\frac{DL}{AD} = \frac{m}{n} = \frac{EL'}{AE}, \quad (1^\circ)$$

$$\therefore \frac{DB}{AD} = \frac{m}{n} + \frac{x}{n}, \text{ and } \frac{EC}{AE} = \frac{m}{n} + \frac{x'}{n},$$

since DB and EC are slightly $> DL, EL'$, resp.

Now when n is taken indefinitely great, $\frac{x}{n}, \frac{x'}{n}$ become indefinitely small,

$$\therefore \frac{DB}{AD} = \frac{EC}{AE}, \quad \text{Q.E.D.} \quad (255)$$

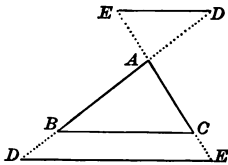
being the limits of variables always equal.

275. COR. *If two sides of a triangle are cut by a parallel to the base, one side is to either of its parts as the other is to its corresponding part.*

For since $DB : AB = EC : AC$, (274)

$\therefore DB + AB : DB$, or AB ,

$$= EC + AC : EC, \text{ or } AC, \quad (246)$$

i.e., $AD : AB$, or DB , $= AE : AC$, or EC . 

SCHOLIUM. It is obvious that the theorems (274, 275) hold good when DE meets AB, AC , produced in either direction.

EXERCISE 352. Circumferences described on the arms of an isosceles triangle as diameters, will intersect in the mid point of the base.

353. Circumferences described on any two sides of a triangle as diameters, will intersect in the third side or the third side produced.

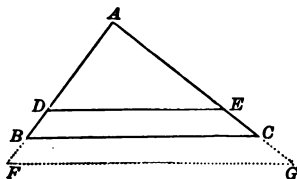
354. If an intersecting circumference pass through the center of another, the angle in the exterior segment of the latter is acute.

355. AB is a common exterior tangent to two circles touching at P ; draw PA, PB ; APB is a right angle.

356. A tangent at the mid point of an arc is parallel to its chord.

PROPOSITION IX. THEOREM.

§ 276. *Conversely, if a straight line divide two sides of a triangle proportionally, that line is parallel to the third side.*



Given: In triangle ABC , DE meeting AB , AC , so that $AD : DB = AE : EC$;

To Prove: DE is parallel to BC .

Produce AB to F so that $BF = DB$, and through F draw $FG \parallel$ to DE to meet AC produced, in G .

Since FG is \parallel to DE , (Const.)

$AD : DF = AE : EG$. (274)

But $AD : DB = AE : EC$, (Hyp.)

$\therefore DB : DF = EC : EG$. (249)

Now $DB = \frac{1}{2} DF$, (Const.)

$\therefore EC = \frac{1}{2} EG$,

$\therefore BC$ is \parallel to DE . Q.E.D. (150)

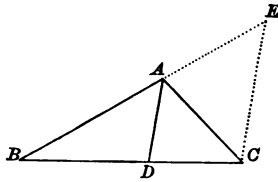
277. DEFINITION. According as a point is taken in a given line, or in that line produced, the distances of the point from the extremities of the given line are called *internal* or *external segments* of the line. Hence the given line is the *sum* of any two internal segments, and the *difference* of any two external segments.



Thus if AB is divided in C and produced to D , then AB is the sum of the internal segments AC , CB , and the difference of the external segments AD , BD .

PROPOSITION X. THEOREM.

278. *The bisector of an interior angle of a triangle divides the opposite side into internal segments having the same ratio as the other two sides.*



Given: In triangle ABC , AD bisecting angle BAC , and meeting BC in D ;

To Prove: $BD : DC = AB : AC$.

Through C draw $CE \parallel$ to AD , to meet BA produced, in E .

Since AD is \parallel to CE , (Const.)

$\angle BAD = \angle E$, (112")

and $\angle CAD = \angle ACE$. (110")

But $\angle BAD = \angle CAD$, (Hyp.)

$\therefore \angle ACE = \angle E$, (Ax. 1)

$\therefore AE = AC$. (65)

Again, since AD is \parallel to CE ,

$BD : DC = BA : AE$. (274)

i.e., $BD : DC = AB : AC$. Q.E.D.

279. COR. *Conversely, in $\triangle ABC$, if D be a point in BC such that*

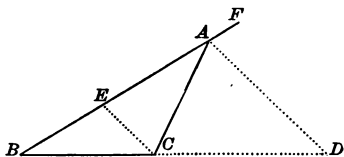
$BD : DC = AB : AC$,

then AD bisects $\angle BAC$.

For the bisector of that angle must pass through D (278), and must therefore coincide with AD .

PROPOSITION XI. THEOREM.

280. *The bisector of an exterior angle of a triangle divides the opposite side into external segments having the same ratio as the other two sides.*



Given: In triangle ABC , AD bisecting exterior angle CAF and meeting BC in D ;

To Prove: $BD : DC = AB : AC$.

Through C draw $CE \parallel$ to AD , to meet BA in E .

Since AD is \parallel to CE , (Const.)

$\angle FAD = \angle AEC$, (112'')

and $\angle CAD = \angle ACE$. (110'')

But $\angle FAD = \angle CAD$, (Hyp.)

$\therefore \angle ACE = \angle AEC$, (Ax. 1)

$\therefore AE = AC$. (65)

Again, since AD is \parallel to CE ,

$BD : DC = BA : AE$, (274)

i.e., $BD : DC = AB : AC$. Q.E.D.

281. COR. *Conversely, in $\triangle ABC$, if D be a point of BC produced, such that*

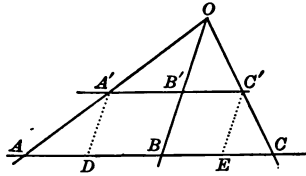
$$BD : DC = AB : AC,$$

then AD bisects the exterior $\angle CAF$.

For the bisector of that angle must pass through D , and therefore must coincide with AD .

PROPOSITION XII. THEOREM.

282. *If three or more transversals passing through the same point make intercepts upon two parallels, these intercepts are proportional.*



Given: Transversals OA , OB , OC , cutting the parallels AC , $A'C'$, in A , B , C , and A' , B' , C' , respectively;

To Prove: $AB : A'B' = BC : B'C'$.

Through A' , C' , draw $A'D$, $C'E$, each \parallel to OB . Then

since $A'C'$ is \parallel to AC , (Hyp.)

$$AO : A'O = BO : B'O = CO : C'O. \quad (274)$$

Since $A'D$, $C'E$, are each \parallel to OB , (Const.)

$$DB = A'B', \text{ and } BE = B'C'; \quad (138)$$

$$\left. \begin{array}{l} \text{also } AB : DB \text{ or } A'B' = AO : A'O, \\ \text{and } BC : BE \text{ or } B'C' = CO : C'O, \end{array} \right\} \quad (274)$$

$$\therefore AB : A'B' = BC : B'C'. \quad \text{Q.E.D.} \quad (232''')$$

As this holds true of the intercepts made by *any* three transversals through O , it holds true of the intercepts made by any number of such transversals.

283. COR. *The parallel intercepts between any two converging transversals are proportional to the intercepts between the parallels and the common point.*

$$\text{For } AB : A'B' = AO : A'O = BO : B'O. \quad (\text{Above})$$

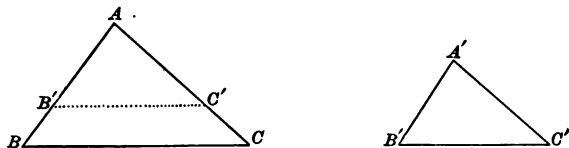
SIMILAR POLYGONS.

284. *Similar polygons* are such as are mutually equiangular, and have the sides about the equal angles, taken in the same order, proportional.

285. In similar polygons, similarly situated points, lines, or angles are said to be *homologous*.

PROPOSITION XIII. THEOREM.

286. *Triangles that are mutually equiangular are similar.*



Given: In triangles ABC , $A'B'C'$, angle A equal to angle A' , angle B to angle B' , and angle C to angle C' ;

To Prove: $BC : B'C' = AB : A'B' = AC : A'C'$.

Place $\triangle A'B'C'$ upon $\triangle ABC$, so that $\angle A' \neq \angle A$, and $\triangle A'B'C'$ takes the position $AB'C'$.

Since $\angle B' = \angle B$, (Hyp.)

$B'C'$ is \parallel to BC , (112')

$\therefore BC : B'C' = AB : AB' = AC : AC'$; (283)

i.e., $BC : B'C' = AB : A'B' = AC : A'C'$. Q.E.D.

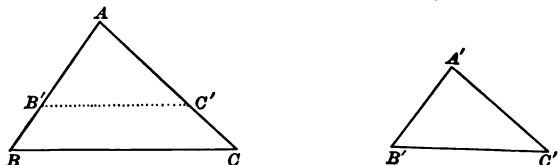
287. COR. 1. *Two triangles are similar, if two angles of the one are respectively equal to two angles of the other.* (121)

288. COR. 2. *Two right triangles are similar, if an acute angle of the one is equal to an acute angle of the other.* (123)

289. SCHOLIUM. In similar triangles, homologous sides lie opposite equal angles.

PROPOSITION XIV. THEOREM.

16. 290. *Two triangles are similar, if an angle of the one is equal to an angle of the other, and the sides about these angles are proportional.*



Given: In triangles ABC , $A'B'C'$, angle A equal to angle A' ; also $AB : A'B' = AC : A'C'$;

To Prove: Triangle ABC is similar to triangle $A'B'C'$.

Place $\triangle A'B'C'$ upon $\triangle ABC$, so that $\angle A' \neq \angle A$, and $\triangle A'B'C'$ takes the position $AB'C'$. Then

since $AB : A'B' = AC : A'C'$, (Hyp.)

$B'C'$ is \parallel to BC , (276)

$\therefore \angle B = \angle B'$, and $\angle C = \angle C'$, (112")

$\therefore \triangle ABC$ is similar to $\triangle A'B'C'$. Q.E.D. (286)

EXERCISE 357. In the diagram for Prop. XI., if the bisector of angle BAC be drawn so as to meet the base in D' , show that BC is divided into internal and external segments having the same ratio.

DEFINITION. — A straight line is said to be divided *harmonically* if it is divided into internal and external segments having the same ratio; or if it is divided into three segments such that the whole line is to either of its outer segments as the other outer segment is to the inner segment.

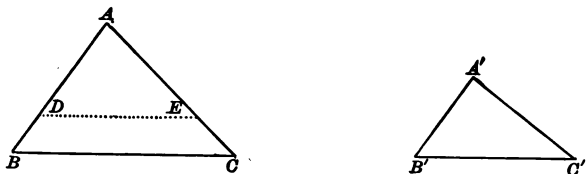
358. Show that in the diagram for Exercise 357, BC is divided harmonically according to the first definition above, and BD , according to the second.

359. Show that if D and D' divide any line MN harmonically according to the first definition, then also M and N divide DD' harmonically.

360. Also show that MD is divided harmonically according to the second definition.

PROPOSITION XV. THEOREM.

15. 291. *Triangles that have their sides mutually proportional are similar.*



Given: In triangles ABC , $A'B'C'$, $AB : A'B' = AC : A'C' = BC : B'C'$;

To Prove: Triangle $A'B'C'$ is similar to triangle ABC .

On AB , AC , take AD , $AE = A'B'$, $A'C'$, resp., and join DE .

Since $AB : AD = AC : AE$, (Hyp. and Const.)

DE is \parallel to BC , (276)

$\therefore \angle D = \angle B$, and $\angle E = \angle C$, (112'')

and $AB : AD = BC : DE$. (283)

But $AB : A'B' = BC : B'C'$. (Hyp.)

$\therefore AD : A'B' = DE : B'C'$. (249)

Now $AD = A'B'$, (Const.)

$\therefore DE = B'C'$;

and $AE = A'C'$, (Const.)

$\therefore \triangle ADE = \triangle A'B'C'$, (69)

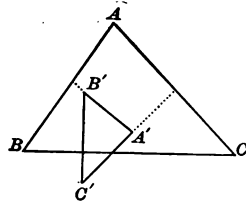
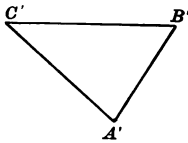
$\therefore \angle B' = \angle D = \angle B$, and $\angle C' = \angle E = \angle C$, (Ax. 1)

$\therefore \triangle A'B'C'$ is similar to $\triangle ABC$. Q.E.D. (286)

SCHOLIUM. From this and Prop. XI. it is seen that mutually equiangular triangles have their homologous sides *proportional*; and conversely.

PROPOSITION XVI. THEOREM.

17. 292. *Two triangles are similar, if they have their sides respectively parallel or perpendicular to each other.*



Given: Triangles ABC , $A'B'C'$, having their sides respectively parallel or respectively perpendicular to each other;

To Prove: Triangle ABC is similar to triangle $A'B'C'$.

Since their sides are \parallel or \perp to each other, (Hyp.) any two homologous angles of these triangles must be either equal or supplementary. (116, 118)

Hence there are three conceivable cases to consider.

1°. Suppose all the angles of the one triangle respectively supplemental to the homologous angles of the other.

i.e., $A + A' = \text{a st. } \angle$, $B + B' = \text{a st. } \angle$, and also $C + C' = \text{a st. } \angle$.

2°. Suppose two supplemental and one equal.

i.e., $A = A'$, $B + B' = \text{a st. } \angle$, and $C + C' = \text{a st. } \angle$.

3°. Suppose all the angles are mutually equal.*

i.e., $A = A'$, $B = B'$, and $\therefore C = C'$.

Each of the first two suppositions must be rejected, since

* If two are right angles, they are equal and supplemental.

For 18 see Phillips & Fisher.

the sum of the angles of two triangles cannot exceed two straight angles. Hence the third alone is admissible;

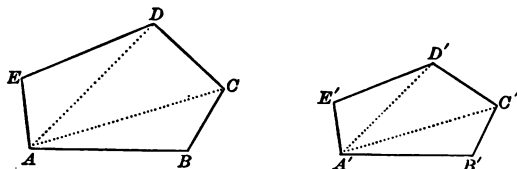
$$\therefore \triangle ABC \text{ is similar to } \triangle A'B'C'. \quad \text{Q.E.D.} \quad (287)$$

293. SCHOLIUM. The homologous sides are those mutually parallel or perpendicular.

—◆—

PROPOSITION XVII. THEOREM.

294. *Two polygons are similar if composed of the same number of triangles similar each to each and similarly placed.*



Given: In polygons P and P' , triangles AED , ADC , ACB , similar to triangles $A'E'D'$, $A'D'C'$, $A'C'B'$, respectively, and similarly placed;

To Prove: P is similar to P' .

1°. P and P' are mutually equiangular.

For their homologous \angle s are either homologous \angle s of similar \triangle s, or are like sums of homologous \angle s of similar \triangle s; (Hyp.)

$$\therefore \angle E = \angle E', \angle EDA + \angle ADC = \angle E'D'A' + \angle A'D'C', \text{ etc.}$$

2°. The homologous sides of P and P' are proportional.

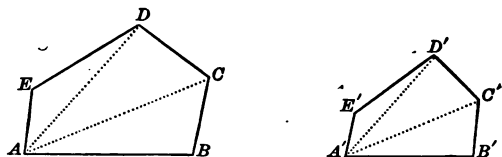
For since the \triangle s are similar, (Hyp.)

$$AB : A'B' = BC : B'C' = AC : A'C' = CD : C'D', \text{ etc.};$$

$$\therefore P \text{ is similar to } P'. \quad \text{Q.E.D.} \quad (284)$$

PROPOSITION XVIII. THEOREM.

295. *Conversely, two similar polygons may be divided into the same number of triangles, similar to each other and similarly placed.*



Given: $ABCDE$, or P , and $A'B'C'D'E'$, or P' , two similar polygons;

To Prove: P and P' may be divided into the same number of similar triangles similarly placed.

From A and A' , homologous vertices of P and P' , draw diagonals AC , AD , and $A'C'$, $A'D'$, respectively.

1°. The number of triangles thus formed in P and P' , respectively, is the same. For it is equal to the number of sides in each, less two.

2°. These triangles are similar and similarly placed.

For since P is similar to P' , (Hyp.)

$$\left. \begin{array}{l} \angle E = \angle E', \\ \text{and } AE : ED = A'E' : E'D', \end{array} \right\} \quad (284)$$

$$\therefore \triangle AED \text{ is similar to } \triangle A'E'D', \quad (290)$$

$$\text{and } \angle EDA = \angle E'D'A'. \quad (289)$$

$$\text{But } \angle EDC = \angle E'D'C', \quad (\text{Hyp.})$$

$$\therefore \angle EDC - \angle EDA = \angle E'D'C' - \angle E'D'A', \quad (\text{Ax. 3})$$

$$\text{i.e., } \angle ADC = \angle A'D'C'.$$

Again, since $\triangle AED$ is similar to $\triangle A'E'D'$, (Above)

$$ED : DA = E'D' : D'A'. \quad (285)$$

$$\text{But } ED : DC = E'D' : D'C', \quad (\text{Hyp.})$$

$$\therefore DA : DC = D'A' : D'C', \quad (249)$$

$$\therefore \triangle ADC \text{ is similar to } \triangle A'D'C'. \quad (290)$$

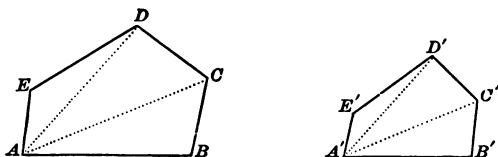
In the same way, the remaining triangles of P may be proved similar to the similarly placed triangles in P' .

$\therefore P$ and P' may be divided as stated. Q.E.D.



PROPOSITION XIX. THEOREM.

296. *The perimeters of two similar polygons have the same ratio as any two homologous sides.*



Given: $AB, A'B'$, any two homologous sides of two similar polygons, P, P' , of which p and p' are the perimeters;

To Prove: $p : p' = AB : A'B'$.

Since P and P' are similar polygons, (Hyp.)

$$AB : A'B' = BC : B'C' = CD : C'D', \text{ etc.,} \quad (284)$$

$$\therefore AB + BC + CD + \dots : A'B' + B'C' + C'D' + \dots = AB : A'B'. \quad (251)$$

But $p = AB + BC + CD + \dots$, and $p' = A'B' + B'C' + C'D' + \dots$,

$$\therefore p : p' = AB : A'B'. \quad \text{Q.E.D.}$$

EXERCISE 361. Draw a diagram to show that two figures may be mutually equiangular though not similar; and another to show that two figures may have their sides mutually proportional and yet not be similar.

362. In the diagram for Prop. XVII., if the numerical measures of BC and $B'C'$ are, respectively, 16 and 10, what is the ratio of p to p' ?

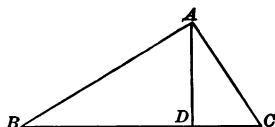
RATIOS OF CERTAIN LINES.

PROPOSITION XX. THEOREM.

297. *In a right triangle, if a perpendicular be drawn from the vertex of the right angle to the hypotenuse,*

1°. The perpendicular is a mean proportional between the segments of the hypotenuse.

2°. Each arm is a mean proportional between the hypotenuse and the adjacent segment.



Given: In a right triangle ABC , AD perpendicular to BC from the right angle BAC ;

To Prove: $\left\{ \begin{array}{l} 1^\circ. \quad BD : DA = DA : DC. \\ 2^\circ. \quad BC : AB = AB : BD, \text{ and } BC : AC = AC : DC. \end{array} \right.$

Since BDA and BAC are rt. Δ , (Hyp.)

and acute $\angle B$ is common to both,
rt. ΔBDA is similar to rt. ΔBAC . (288)

Since CDA and BAC are rt. Δ , (Hyp.)

and acute $\angle C$ is common to both,
rt. ΔCDA is similar to rt. ΔBAC , (288)

\therefore rt. ΔBDA is similar to rt. ΔCDA , (286)

each being similar to rt. ΔBAC .

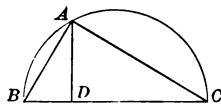
Since ΔBDA is similar to ΔCDA ,

$$BD : DA = DA : DC. \quad \text{Q.E.D.} \quad (284)$$

Since ΔBAC is similar to ΔBDA and CDA ,

$$\left. \begin{array}{l} BC : AB = AB : BD, \\ \text{and } BC : AC = AC : DC. \end{array} \right\} \quad \text{Q.E.D.} \quad (284)$$

298. COR. *If from any point A in a circumference, a perpendicular be drawn to a diameter BC, and chords AB, AC, be drawn,*



since BAC is a right angle, (267)

$$DC : DA = DA : DB,$$

BC : BA = BA : BD, and BC : CA = CA : CD.

} (297)

Hence

1°. *The perpendicular from any point of a circumference to a diameter is a mean proportional between the segments.*

2°. *The chord drawn from the point to either extremity of the diameter is a mean proportional between the diameter and the adjacent segment.*

299. DEFINITION. Two ratios are said to be mutually *inverse* or *reciprocal* when the antecedent and the consequent of the one are, respectively, the consequent and the antecedent of the other. Thus $B : A$ is the inverse of $A : B$, and $11 : 7$ is the inverse of $7 : 11$.

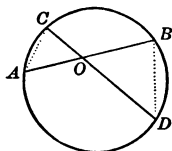
300. DEFINITION. If four quantities, A, B, C, D , are so related that

$$A : C = D : B, \text{ or } \frac{A}{C} = \frac{D}{B},$$

i.e., if the first has to the third the inverse ratio of the second to the fourth, the quantities are said to be *inversely proportional*, while, as we know, if the first is to the third as the second to the fourth, the quantities are directly proportional. As will be seen in the next proposition, two lines have their segments *inversely* proportional if a segment of the first is to a segment of the second as the remaining segment of the second is to the remaining segment of the first; while the segments would be *directly* proportional if a segment of the first were to a segment of the second as the remaining segment of the first to the remaining segment of the second.

PROPOSITION XXI. THEOREM.

301. *If two chords intersect, their segments are inversely proportional.*



Given: In circle ADB , chords AB , CD , intersecting in O ;

To Prove: $OA : OD = OC : OB$.

Join AC and BD . Then

since $\angle A = \angle D$, and $\angle B = \angle C$, (264)

$\triangle AOC$ is similar to $\triangle DOB$, (286)

$\therefore OA : OD = OC : OB$. Q.E.D. (289)

SCHOLIUM. In deducing such proportions between pairs of sides in similar triangles, the student may find it useful to remember that, as the homologous sides are opposite equal angles, *the greater side of the one is to the greater side of the other as the lesser side of the one is to the lesser side of the other.*

EXERCISE 363. In the diagram for Prop. XI., if a parallel to BC cut AB , AC , and AD , in G , H , and L , respectively, show that AD is divided by GL so that $AL : AD = GH : BC$.

364. In the diagram for Prop. XII., show that the triangles $A'AD$ and $OA'B'$ are similar to each other and to triangle OAB .

365. In the same diagram, show that the perimeter of triangle OAB is to that of triangle $OA'B'$ as OB is to OB' .

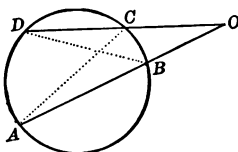
366. In the diagram for Prop. XX., if $\angle B : \angle C = 3 : 5$, how many degrees in each of the acute angles at A ?

367. In the same diagram, if the numerical measures of BC and DC are 10 and 2, respectively, what is the numerical measure of AD ?

368. Find also the numerical measures of AB and AC .

PROPOSITION XXII. THEOREM.

302. If two secants be drawn from a point without a circle, these secants and their external segments are inversely proportional.



Given: Two secants OA, OD , cutting circle ADB in B, A , and C, D , respectively;

To Prove: $OA : OD = OC : OB$.

Join AC and BD . Then

since $\angle A = \angle D$, and $\angle O$ is common to both, (264)

$\triangle AOC$ is similar to $\triangle DOB$, (287)

$\therefore OA : OD = OC : OB$. Q.E.D. (289)

EXERCISE 369. Prove Prop. XXI. by joining AD and BC , and showing that $OA : OD = OC : OB$.

370. In the diagram for Prop. XXII., show that the triangles whose vertices are at the intersection of AC and BD are similar.

371. If two equal chords intersect, their segments are severally equal.

372. If equal chords be produced to meet, the secants thus formed and their external segments will be severally equal.

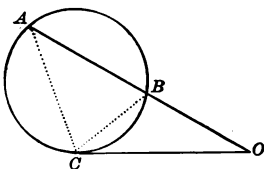
373. In the diagram for Prop. XXI., show that if the chords are equally distant from the center, they are *directly* as well as *inversely* proportional.

374. In the diagram for Prop. XXII., show that if the secants are equally distant from the center, they are *directly* as well as *inversely* proportional.

375. If two circles intersect, tangents drawn to them from any point in their common chord produced, will be equal.

PROPOSITION XXIII. THEOREM.

303. *If a secant and a tangent be drawn from a point without a circle, the tangent is a mean proportional between the secant and its external segment.*



Given: A tangent OC touching circle ABC in C , and a secant OA cutting ABC in B, A ;

To Prove: $OA : OC = OC : OB$.

Join AC and BC . Then

$\therefore \angle A$ and $\angle OCB$ are each meas. by $\frac{1}{2}$ arc BC , (264, 269)

$$\angle A = \angle OCB,$$

and $\angle O$ is common to $\triangle OAC$ and OBC ,

$\therefore \triangle OAC$ is similar to $\triangle OBC$, (287)

$\therefore OA : OC = OC : OB$. Q.E.D. (289)

EXERCISES.

QUESTIONS.

376. In the diagram for Prop. VIII., if $DB : AD = 3 : 7$, and $AC = 55$, what is the value of AE ?*

377. If in the preceding question we substitute $\sqrt{11}$ for 3, what is the value of AE ?

* *Value*, here and elsewhere, stands for numerical measure, the unit being left undetermined. If decimals occur in a result, it will be sufficient to have two places correct.

378. In the diagram for Prop. X., if $AB, AC, BC, = 9, 7, 12$, respectively, what is the value of BD and of DC ?

379. In the diagram for Prop. XI., if $AB, AC, BD, = 9, 7, 20$, respectively, what is the value of BC ?

380. If the angles at the base of $\triangle ABC$ in Prop. XI. are equal, how is the proposition modified?

381. If two triangles have an angle of the one equal to an angle of the other, and the sides about another angle proportional, are they necessarily similar?

382. In the diagram for Prop. XII., if $AB, AC, A'B', = a, b, a'$, respectively, what is the value of $A'C'$?

383. In the diagram for Prop. XVII., if $AD, DE, AC, A'D', = 4, 3, 5, 3.2$, respectively, what are the values of $D'E', A'C'$?

384. In the diagram for Prop. XX., if $AB, AC, BC, = 4, 3, 5$, respectively, what are the values of AD, BD, DC ?

385. In the diagram for Prop. XXI., if $AO = \frac{1}{2} OD$, and $OB = 8$, what is the value of OC ?

386. In the diagram for Prop. IX., if BD is $\frac{1}{m}$ of AD , what part is CG of AG ?

387. In the diagram for Prop. X., if $AB : AC, = 10 : 7$, what is the ratio of AD to EC ?

388. In the same diagram, if $AB = AC$, how many degrees are there in angle BCE ?

389. In the diagram for Prop. XI., if $EC : AD, = 2 : 3$, what is the ratio of AB to AC ?

390. In the same diagram, if AB is equal to AC , where will the point E fall?

391. In the diagram for Prop. XII., if $AC = 12$ and $OB : OB' = 8 : 5$, what is the value of $AD + EC$?

392. In the same diagram, if OB bisects angle AOC , $AB = 10$, and $BC = 8$, what is the ratio of OA' to OC' ?

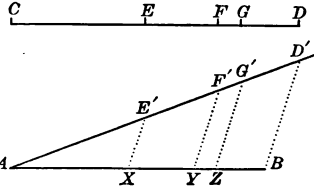
393. In the diagram for Prop. XIII., if $BC : B'C' = m : n$, what is the ratio of $AB - AC$ to $A'B' - A'C'$?

394. In the diagram for Prop. XVII., if $AB = 18$, and $A'B' = 12$, what is the ratio of AC to $A'C'$?

395. In the diagram for Prop. XX., if $BD : DA = m : n$, what is the ratio of the perimeter of triangle ADB to the perimeter of triangle ADC ?

CONSTRUCTIONS.

304. To divide a given line AB proportionally to another given line CD divided in E, F, G .

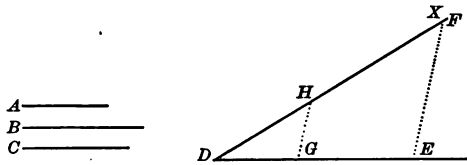


Upon AD' , making any angle with AB , lay off AE' , $E'F'$, $F'G'$, $G'D'$, $= CE, EF, FG, GD$, respectively. Join BD' , and through E', F', G' , draw lines parallel to BD' , meeting AB in X, Y, Z , respectively. Then (274),

$$AX : XY : YZ : ZB = AE' : E'F' : F'G' : G'D' = CE : EF : FG : GD.$$

If instead of a divided line we have numbers given, say 3, 7, 9, etc., we lay off on AD' , $AE' = 3$, $E'F' = 7$, etc., and AB will be divided proportionally to the given numbers.

305. To find a fourth proportional to three given lines A, B, C .



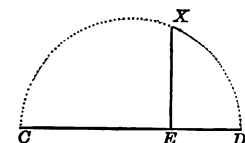
Draw DE, DF , making any angle with each other. Upon DE lay off DG, GE , equal to A and B , respectively, and on DF lay off $DH = C$. Join GH , and through E draw EX parallel to GH , and meeting DF in X . Then

$$DG : GE = DH : HX, \text{ or } A : B = C : HX. \quad (274),$$

It is obvious that if $B = C$, we take $GE = DH$, and we obtain by this construction a third proportional to A and B .

306. To find a mean proportional between two lines, A , B .

Upon an indefinite line lay off ED , CE , respectively equal to A and B . Upon CD as diameter describe a semicircle DXC ; at E draw EX perpendicular to CD to meet the circumference in X . Then (312"),



ED or $A : EX = EX : CE$ or B .

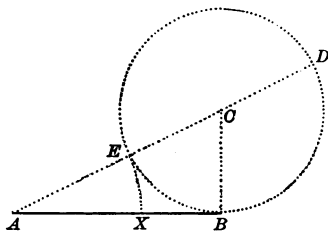
A _____
 B _____

307. DEFINITION. A straight line is said to be divided in *extreme and mean ratio*, when it is divided into two segments such that the greater segment is a mean proportional between the whole line and the lesser segment. Thus if AB is divided in C so that $AB : AC = AC : BC$, then AB is divided in extreme and mean ratio.



308. To divide a line AB in extreme and mean ratio.

At B draw $BC \perp$ to AB , and make $BC = \frac{1}{2} AB$.



From C as center, with radius CB , describe a circumf. BDE .

Through A and C draw a secant meeting the circumference in D and E .

From A as center, with radius AE , describe an arc EX meeting AB in X .

Then $AB : AX = AX : BX$. For

$\therefore AB$ is a tangent (191), and AD a secant to $\odot BDE$, (Const.)

$$AE : AB = AB : AD, \quad (303)$$

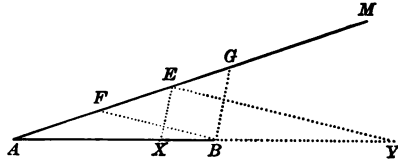
$$\therefore AE : AB - AE = AB : AD - AB. \quad (247)$$

But $AB = 2 BC = ED$, whence $AD - AB = AE = AX$,

$$\therefore AX : BX = AB : AX, \text{ or } AB : AX = AX : BX. \quad (307)$$

309. To divide a given line AB harmonically in the ratio of two lines C and D .

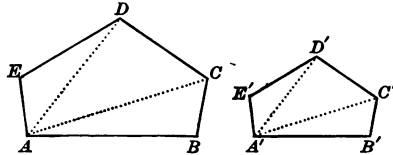
Upon the line AM , making any angle with AB , lay off $AE = C$; and on each side of E lay off EF, EG , each equal to D . Join FB, GB , and draw EX, EY , parallel to GB and FB , respectively. Then,



$$\left. \begin{aligned} AX : BX &= AE : EG = C : D, \\ \text{and } AY : BY &= AE : FE = C : D. \end{aligned} \right\} \quad (275)$$

310. Upon a straight line $A'B'$ to construct a polygon similar to a given polygon.

Divide the given polygon $ABCDE$ into triangles by diagonals drawn from vertex A . At the extremities of

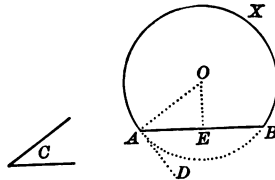


$A'B'$, make $\angle B' = \angle B$, and $\angle B'A'C' = \angle BAC$. Then will $\triangle A'B'C'$ be similar to $\triangle ABC$ (288). In like manner construct $\triangle A'C'D'$ similar to $\triangle ACD$, and $\triangle A'D'E'$ similar to $\triangle ADE$.

Polygon $A'B'C'D'E'$ is similar to polygon $ABCDE$. (294)

311. Upon a given line AB , to describe a segment of a circle such that any angle inscribed in it shall equal a given $\angle C$.

At A , make $\angle BAD = \angle C$ (203); draw AO perpendicular to AD at A , and draw EO perpendicular to AB at its mid point E . From O the intersection of AO, EO , describe arc ABX . ABX is the required segment. (269)

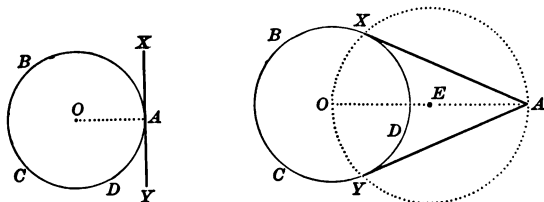


Proof. $\angle BAD = \angle C$, is measured by $\frac{1}{2}$ arc ADB , as is also any angle inscribed in BXA .

312. From a given point A , in or without a given circumference BCD , to draw a tangent to BCD .

Find O , the center of BCD , (173) and join OA . Then

1°. If A is in the circumference, draw XY perpendicular to OA at A . XY is tangent to BCD at A . (191)



2°. If A is without the circumference, bisect OA in E . From E as center, with radius EA , describe a circumference AXY , cutting BCD in X and Y .

Join AX, AY ; then AX, AY are tangents to BCD . (267, 191)

Proof. Join OX, OY , and show that OX, OY are perpendicular to AX, AY , respectively.

EXERCISES.

THEOREMS.

396: The chords that join the near extremities of equal chords are parallel.

397. The opposite angles of a quadrilateral inscribed in a circle are supplementary.

DEFINITION. Three or more points are said to be *concylic* if a circumference can be described through them. Thus the preceding theorem could be enunciated thus: A quadrilateral whose vertices are concyclic has its opposite angles supplementary.

398. If two opposite angles of a quadrilateral are supplementary, its vertices are concyclic.

399. If AB, CD , intersect in O , so that $AO : OC = OD : OB$, then A, B, C, D , are concyclic.

400. If OA, OD , are divided in B and C , respectively, so that $OA : OD = OC : OB$, then A, B, C, D , are concyclic.

401. If an arc be divided into three equal parts by chords drawn from one extremity of the arc, the middle chord bisects the angle formed by the other two.

402. If, from any point of a circumference, a tangent and a chord be drawn, the perpendiculars upon these lines from the mid point of the intercepted arc are equal.

403. The diagonals of a trapezoid cut each other in the same ratio.

404. If through one of the points of intersection of two equal circles, any line be drawn to meet the circumferences, the extremities of this line are equally distant from the other point of intersection.

405. If, in a right triangle, the altitude upon the hypotenuse divides it in extreme and mean ratio, the lesser arm is equal to the farther segment.

406. In any right triangle, one arm is to the other as the difference of the hypotenuse and the second arm is to the intercept on the first arm between the right vertex and the bisector of the opposite acute angle.

407. The altitudes of a triangle are inversely proportional to the sides upon which they are drawn.

408. If from an angle of a parallelogram $ABCD$, a line be drawn cutting a diagonal in E and the sides in P and Q , respectively, then will AE be a mean proportional between PE and QE .

409. If from the extremities of a diameter, perpendiculars be drawn to any chord of the circle, the feet of these perpendiculars will be equally distant from the center.

410. Show that there may be two, but not more than two, similar triangles in the same segment of a circle.

411. If two circles are tangent externally, a common exterior tangent is a mean proportional between their diameters.

412. The chord drawn from the vertex of an inscribed equilateral triangle to any point in the opposite arc is equal to the sum of the chords drawn to that point from the other vertices.

413. If two circles are tangent externally, lines drawn through the point of contact to the circumferences are divided proportionally at the point of contact.

414. If a circle is tangent to another internally, all chords of the outer circle drawn from the point of contact are divided proportionally by the circumference of the inner circle.

415. Chords drawn from the point of contact of a tangent have their segments made by any chord parallel to the tangent, inversely proportional.

416. If a tangent is intercepted between two parallel tangents to the same circle, its segments made by the point of contact have the radius as mean proportional.

417. If three circles intersect each other, their three common chords pass through the same point.

418. If through the mid point of a side of a triangle a line be drawn intersecting a second side, the third side produced, and a line parallel to the first through the opposite vertex, the line will be divided harmonically.

PROBLEMS.

419. To a given circle draw a tangent that shall be perpendicular to a given line.

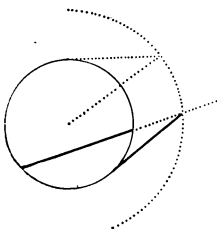
420. To a given circle draw a tangent that shall be parallel to a given line.

421. To a given circle draw two tangents including a given angle.

422. In a given straight line, find a point such that the tangents drawn from it to a given circle shall include the greatest angle.

423. In a chord produced, find a point such that the tangent from that point shall be equal to a given line.

424. From a given center describe a circle tangent to a given circle.



425. Describe a circumference passing through a given point and touching a given circle in a given point.

426. Describe two circles with given radii, so intersecting that their common chord shall have a given length not greater than the lesser diameter.

427. With a given radius, describe a circle tangent to two given circles.

428. Draw a common exterior tangent to two given circles.

429. Draw a common interior tangent to two given circles.

430. Find a point such that the tangents drawn from it to the outer sides of two tangent circles shall include a given angle.

431. Divide any side of a triangle into two parts proportional to the other sides.

432. Divide any side of a triangle into three parts proportional to the three sides.

433. From a given line cut off a part that shall be a mean proportional between the remainder and another given line.

434. Through a given point within a circle, draw a chord there divided in the same ratio as a given chord through that point.

435. From a given point without a circle draw a secant divided by the circumference in a given ratio.

436. From a given point in a given arc draw a chord bisected by the chord of the given arc.

437. In a given circle place a chord that shall be trisected by two given radii at right angles to each other.

438. In a given circle place a chord parallel to a given chord, and having to it a given ratio not greater than that of the diameter to the given chord.

439. Through one of the points of intersection of two given circles draw a secant forming chords that are in a given ratio.

440. Inscribe a square in a given triangle.

441. Inscribe a square in a given segment of a circle.

442. In a given semicircle inscribe a rectangle similar to a given rectangle.

443. In a given circle inscribe a triangle similar to a given triangle.

444. About a given circle circumscribe a triangle similar to a given triangle.

445. In a given triangle construct a parallelogram similar to a given parallelogram.

446. Construct a triangle having given the base, the vertical angle, and the length of the bisector of that angle.

LOCI.

447. Find the locus of the center of each circumference that passes through two given points.

448. Find the locus of the center of each circle that is tangent to a given circle at a given point.

449. Find the locus of the center of each circle of given radius that is tangent to a given circle.

450. Find the locus of the center of each circle that is tangent to a given line at a given point.

451. Find the locus of the center of each circle that is tangent to two given intersecting lines.

452. Find the locus of the points from which pairs of tangents of a given length may be drawn to a given circle.

453. Find the locus of the mid point of any chord that passes through a given point in a given circle.

454. Find the locus of the mid point of any secant that can be drawn from a given point to a given circumference.

455. Find the locus of the vertex of any triangle constructed on a given base, with a given vertical angle.

456. Find the locus of a point whose distances from two given points are in a given ratio.

457. Find the locus of a point whose distances from two given straight lines are in a given ratio.

458. Find the locus of a point the sum of whose distances from two given straight lines is equal to a given line.

459. Find the locus of a point the difference of whose distances from two given straight lines is equal to a given line.

460. Find the locus of the points that divide the chords of a given circle so that the rectangle of their segments is equal to a given square.

BOOK V.

AREAS AND THEIR COMPARISON.



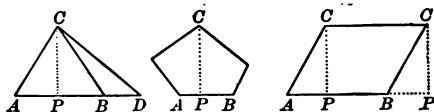
QUADRILATERALS.

313. The *area* of a plane figure is the quantity of its surface as measured by the unit of surface, or is the numerical measure of that quantity.

314. Figures that are not similar but have equal areas are said to be *equivalent*.

315. The *base* of a polygon is any side on which we choose to regard it as constructed.

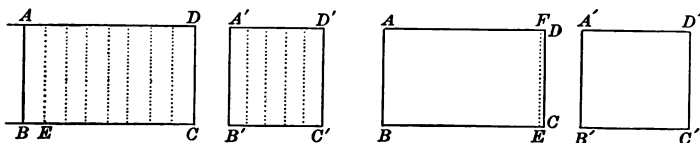
316. The *altitude* of a polygon is the perpendicular distance to the base from the remotest vertex or from a side parallel to the base or the base produced.



Thus in each of the figures above, the perpendicular CP is the altitude of the figure when AB is taken as base. It is obvious that two triangles having their bases in the same line and the opposite vertex common, as ACB and ACD , have the same altitude.

PROPOSITION I. THEOREM.

317. *Rectangles with equal altitudes are to each other as their bases.*



Given: Two rectangles AC , $A'C'$, with equal altitudes AB , $A'B'$, and bases BC , $B'C'$;

To Prove: Rectangle AC : rectangle $A'C' = BC : B'C'$.

1°. When BC and $B'C'$ are commensurable.

Let BE be a common measure of BC and $B'C'$, so that BE can be laid off 5 times on $B'C'$ and 8 times on BC .

Through each point of division draw perpendiculars to the opposite side of the rectangle. The figures thus formed are equal rectangles. (144)

Since BC and $B'C'$ contain 8 and 5 parts, respectively, each equal to BE , (Const.)

AC and $A'C'$, resp., contain 8 and 5 parts, each equal to AE .

$$\therefore BC : B'C' = 8 : 5, \text{ and } AC : A'C' = 8 : 5; \quad (225)$$

$$\therefore \text{rect. } AC : \text{rect. } A'C' = BC : B'C'. \quad \text{Q.E.D. } (232''')$$

2°. When BC and $B'C'$ are incommensurable.

Suppose $B'C'$ divided into any number of equal parts n , and that BC contains this n th part of $B'C'$ m times with a remainder EC . Draw EF perpendicular to BC .

Since BE and $B'C'$ are commensurable, (Const.)

$$\frac{BE}{B'C'} = \frac{m}{n} = \frac{\text{rect. } AE}{\text{rect. } A'C'}, \quad (1^\circ)$$

$$\therefore \frac{BC}{B'C'} = \frac{m}{n} + \frac{x}{n}, \text{ and } \frac{\text{rect. } AC}{\text{rect. } A'C'} = \frac{m}{n} + \frac{x'}{n},$$

since BC and AC are slightly $> BE$ and AE , respectively.

Now, when n is taken indefinitely great, $\frac{x}{n}$ and $\frac{x'}{n}$ become indefinitely small;

$$\therefore \frac{\text{rect. } AC}{\text{rect. } A'C'} = \frac{BC}{B'C'}, \quad \text{Q.E.D.} \quad (236)$$

being the limits of variables always equal.

318. COR. 1. *Rectangles with equal bases are to each other as their altitudes.*

For (317) the equal bases may be taken as altitudes, and the altitudes as bases.

319. SCHOLIUM. Since a rectangle is *determined* by its base and altitude (144), that is, by any two adjacent sides, as AB, BC , we employ the expression, *the rectangle contained by AB and BC* , or more briefly, *the rectangle AB, BC* , to denote the rectangle determined by AB and BC ; and use the symbol $\text{rect. } AB \cdot AC$, or simply $AB \cdot BC$, for either of these expressions. Since a square, again, is determined by its base, *i.e.*, by a side, we employ the expression, *the square of AB* , or the symbol \overline{AB}^2 , an abbreviation for $AB \cdot AB$, to denote the square whose base is AB .

320. DEFINITION. The symbol \approx , to be read, *is equivalent to, is the symbol of equivalence*.

321. COR. 2. *If four lines, A, B, C, D , are in proportion, the rectangle contained by the extremes is equivalent to that contained by the means.*

$$\left. \begin{aligned} \text{For since } \text{rect. } A \cdot D : \text{rect. } B \cdot D &= A : B, \\ \text{and } \text{rect. } B \cdot C : \text{rect. } B \cdot D &= C : D = A : B, \end{aligned} \right\} \quad (317)$$

$$\text{rect. } A \cdot D : \text{rect. } B \cdot C = \text{rect. } B \cdot D : \text{rect. } B \cdot D, \quad (232''')$$

$$\therefore \text{rect. } A \cdot D \approx \text{rect. } B \cdot C.$$

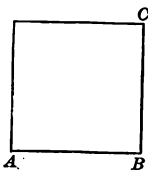
322. COR. 3. *If A, B, C , are lines such that*

$$A : B = B : C,$$

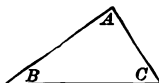
$$\text{then } \overline{B}^2 = \text{rect. } A \cdot C. \quad (321)$$

That is, the square of a mean proportional between two lines is equivalent to the rectangle contained by those lines.

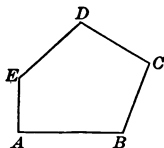
323. The *unit of area* is the square having as base the linear unit. Thus if the base AB of the square AC is equal to the linear unit, then the square AC is the unit with which all areas are compared.



324. As already defined, the numerical measure of a quantity is the number that shows how many times the quantity contains its unit; in other words, it is the *ratio* of the quantity to its unit. As regards triangles, it is customary to denote the numerical measure of a side by means of the small letter corresponding to the capital designating the opposite angle. Thus in $\triangle ABC$, we employ a, b, c , to denote the numerical measures of BC, AC, AB , respectively.



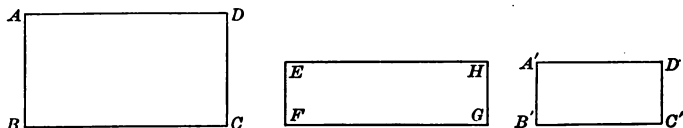
As regards polygons of more sides than three, there is no such convention, but we specify $AB = a, BC = b, CD = c$, and so on. Wherever it may occur, henceforth, such an expression as *the product of A and B* is to be understood as a convenient abbreviation for *the product of the numerical value of A by that of B* . Great care should be taken, however, not to forget the real meaning of such abbreviations. Beginners are often confused



by the careless use of such expressions as, *length multiplied by breadth gives area*, forgetting that what is meant is: *the numerical measure of the length multiplied by that of the breadth gives as result the numerical measure of the area*, as we find explained in Art. 326.

PROPOSITION II. THEOREM.

325. Rectangles are to each other as the products of the numerical measures of their altitudes and bases.



Given: Two rectangles AC , $A'C'$, with altitudes AB , $A'B'$, and bases BC , $B'C'$, respectively, whose numerical measures are, respectively, a , a' , and b , b' ;

To Prove: Rectangle AC : rectangle $A'C' = a \times b : a' \times b'$.

Construct a rectangle EG with altitude $EF = A'B'$, and base $FG = BC$, and let the numerical measures of AC , $A'C'$, EG , be x , y , z , respectively.

Since AC and EG are rectangles with equal bases, (Const.)

$$AC : EG = AB : EF, \text{ or } x : z = a : a'; \quad (318, 232')$$

$\therefore EG$ and $A'C'$ are rectangles with equal altitudes, (Const.)

$$EG : A'C' = FG : B'C', \text{ or } z : y = b : b'. \quad (317, 232')$$

From these numerical proportions we obtain (242)

$$x : y = a \times b : a' \times b',$$

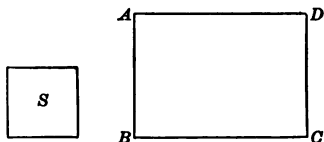
$$\therefore \text{rect. } AC : \text{rect. } A'C' = a \times b : a' \times b' \quad \text{Q.E.D.} \quad (232'')$$

326. Cor. The area of a rectangle is measured by the product of its base and altitude.

For if AC be any rectangle, and S the unit-square, then

$$\therefore \text{area } AC : S = a \times b : 1 \times 1,$$

$$\text{area } AC = S \times ab;$$



i.e., the area of AC is ab

times the unit-square S . If, for example, the numerical measures of AB and BC are 5 and 7, respectively, then the

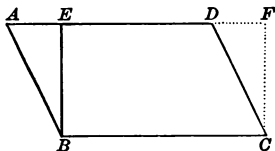
numerical measure of the area of AC is 35; that is, the area of AC is equal to 35 unit-squares.

In the enunciation of this corollary, as elsewhere, the term *area* is for brevity used for *numerical measure of the area*; that is, the number of unit-squares to which the surface in question is equivalent.

—◆—

PROPOSITION III. THEOREM.

327. *Any parallelogram is equivalent to the rectangle having the same base and altitude.*



Given: A parallelogram AC and a rectangle EC , with the same base and altitude BC , EB ;

To Prove: Parallelogram AC is equivalent to rectangle EC .

Since AC and EC are parallelograms, (Hyp.)

$AB = DC$, and $BE = CF$, (136)

\therefore rt. $\triangle ABE =$ rt. $\triangle DCF$. (72)

But $ABCF - \triangle ABE \approx$ rect. EC ,

and $ABCF - \triangle DCF \approx$ par'm AC ,

\therefore par'm $AC \approx$ rect. EC . Q.E.D. (Ax. 3)

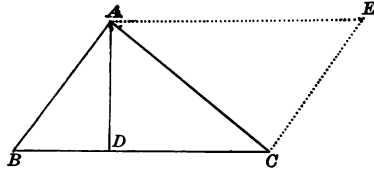
328. COR. 1. *Parallelograms with equal bases and equal altitudes are equivalent.* For each is equivalent to the same rectangle. (327)

329. COR. 2. *Parallelograms with equal altitudes are to each other as their bases; and those with equal bases to each other as their altitudes.* For they are as the rectangles having those bases or altitudes.

330. COR. 3. *The area of any parallelogram is equal to the product of its base and altitude.*

PROPOSITION IV. THEOREM.

331. *Any triangle is equivalent to one half the rectangle contained by its base and altitude.*



Given: A triangle ABC , having a base BC and altitude AD ;

To Prove: Triangle ABC is equivalent to $\frac{1}{2}$ rect. $AD \cdot BC$.

Complete the parallelogram $ABCE$.

Then since AC is a diagonal of par'm BE ,

$$\triangle ABC = \triangle ACE, \quad (140)$$

$$\therefore \triangle ABC \approx \frac{1}{2} \text{ par'm } BE,$$

$$\therefore \triangle ABC \approx \frac{1}{2} \text{ rect. } AD \cdot BC. \quad \text{Q.E.D.} \quad (327)$$

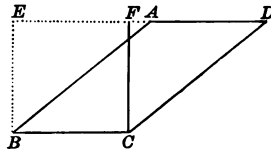
332. COR. 1. *Triangles with equal bases and equal altitudes are equivalent.*

333. COR. 2. *Triangles with equal altitudes are to each other as their bases; and those with equal bases, as their altitudes.*

334. COR. 3. *Triangles are to each other as the products of their bases and altitudes.*

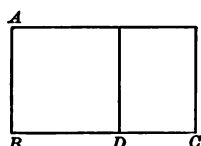
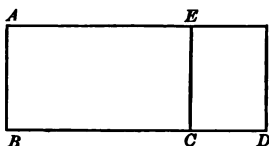
EXERCISE 461. Prove Prop. III., when the upper base of the rectangle lies without that of the given parallelogram, as in the accompanying diagram.

462. Prove the same proposition when the upper and lower bases lie without each other, though in the same lines.



PROPOSITION V. THEOREM.

335. *The rectangle contained by a line and the sum or difference of other two lines, is equivalent to the sum or difference of the several rectangles contained by that line and the other two lines.*



Given: A line AB and another line BD equal to $BC \pm CD$;

To Prove: $\text{rect. } AB \cdot BD \approx \text{rect. } AB \cdot BC \pm \text{rect. } AB \cdot CD$.

Suppose the rectangles $AB \cdot BD$, etc., duly constructed. (206)

1°. Let $BD = BC + CD$. Then, in left-hand Fig.,

$$\text{rect. } AD = \text{rect. } AC + \text{rect. } ED, \quad (\text{Ax. 9})$$

$$\text{i.e., rect. } AB \cdot BD = \text{rect. } AB \cdot BC + \text{rect. } EC \cdot CD, \quad (319)$$

$$\therefore \text{rect. } AB \cdot BD \approx \text{rect. } AB \cdot BC + \text{rect. } AB \cdot CD, \quad \text{q.e.d.}$$

$$(\text{since } EC = AB). \quad (136)$$

2°. Let $BD = BC - CD$. Then, in right-hand Fig.,

$$\text{since rect. } AB \cdot BD + \text{rect. } AB \cdot CD \approx \text{rect. } AB \cdot BC, \quad (1^\circ)$$

$$\text{rect. } AB \cdot BD \approx \text{rect. } AB \cdot BC - \text{rect. } AB \cdot CD. \quad \text{q.e.d. (Ax. 3)}$$

SCHOLIUM. Let a, b, c , denote the numerical measure of AB, BC, CD , respectively; then by substitution we obtain the well-known algebraic formula

$$a(b \pm c) = ab \pm ac.$$

336. COR. 1. *The square of the sum or difference of two lines is equivalent to the sum of the squares of those lines plus or minus twice their rectangle.*

Let A and B be the lines. Then

$$(A \pm B)^2 \approx \text{rect. } (A \pm B) \cdot (A \pm B), \quad (319)$$

$$\left. \begin{aligned} &\approx \text{rect. } A(A \pm B) \pm \text{rect. } B(A \pm B) \\ &\approx A^2 \pm \text{rect. } A \cdot B \pm \text{rect. } B \cdot A + B^2 \\ &\approx A^2 + B^2 \pm 2 \text{ rect. } A \cdot B. \end{aligned} \right\} \quad (335)$$

SCHOLIUM. This result may be expressed algebraically thus: $(a \pm b)^2 = a^2 \pm 2ab + b^2$.

337. COR. 2. *The rectangle contained by the sum and difference of two lines is equivalent to the difference of the squares of the lines.*

Let A and B be the lines. Then

$$\left. \begin{aligned} (A + B) \cdot (A - B) &\approx A(A - B) + B(A - B), \\ &\approx A^2 - A \cdot B + B \cdot A - B^2, \\ &\approx A^2 - B^2. \end{aligned} \right\} \quad (335)$$

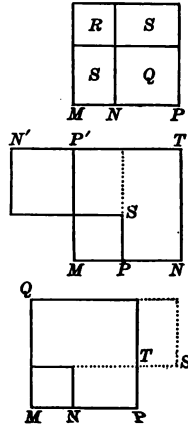
SCHOLIUM. This result may be expressed algebraically thus: $(a + b)(a - b) = a^2 - b^2$.

The above corollaries, pure deductions from Prop. V., become obvious to inspection in the accompanying diagrams.

In the first, we see that the square of MP , the sum of MN and NP , is made up of the squares Q and R and the two equal rectangles S , S .

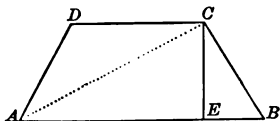
In the second, we see that the square of MP , the difference of MN and NP , is less than $NMN'T$, the sum of the squares of MN and NP , by the figure NSN' , which $= 2PT = 2 \text{ rect. } MN \cdot NP$.

In the third, we see that if from the square of MP we take the square of MN , we obtain the figure QTP , which is equivalent to QS or $(MP + PN) \cdot (MP - PN)$, where $PN = MP - MN$.



PROPOSITION VI. THEOREM.

338. *A trapezoid is equivalent to the rectangle contained by its altitude and half the sum of its parallel sides.*



Given : A trapezoid $ABCD$, with bases AB , CD , and altitude CE ;

To Prove : $ABCD$ is equivalent to rect. $CE \cdot \frac{1}{2}(AB + CD)$.

Join AC .

Since $ABCD \simeq \triangle ABC + \triangle ADC$, (Ax. 9)

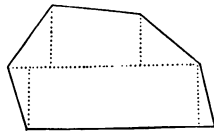
but $\triangle ABC \simeq \frac{1}{2} \text{ rect. } AB \cdot CE$,
and $\triangle ADC \simeq \frac{1}{2} \text{ rect. } CD \cdot CE$, } (331)

$ABCD \simeq \text{rect. } CE \cdot \frac{1}{2}(AB + CD)$. Q.E.D. (335)

SCHOLIUM 1. The above proposition may be expressed under the form :

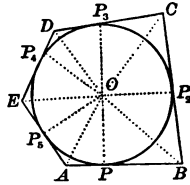
The area of a trapezoid is measured by the product of its altitude by half the sum of its bases.

339. SCHOLIUM 2. The area of any polygon may be found by dividing it into triangles, and finding the areas of these (331). But the method generally employed is to draw the most convenient diagonal of the figure, and draw to it perpendiculars from the other angular points. The figure is thus divided into right triangles, rectangles, or trapezoids, the areas of which are easily found.



PROPOSITION VII. THEOREM.

340. *Any circumscribed polygon is equivalent to the rectangle contained by half its perimeter and the radius of the inscribed circle.*



Given: A polygon $ABCDE$ circumscribed about a circle with radius OP ;

To Prove: $ABCDE \approx \text{rect. } OP \cdot \frac{1}{2}(AB + BC + CD + DE + EA)$.

Join OA , OB , etc., and draw $OP_2 \perp$ to BC , $OP_3 \perp$ to CD , etc.

$$\left. \begin{aligned} \text{Since } \triangle AOB &\approx \frac{1}{2} \text{ rect. } OP \cdot AB, \\ \text{and } \triangle BOC &\approx \frac{1}{2} \text{ rect. } OP_2 \cdot BC, \\ \text{and similarly of the other triangles,} \end{aligned} \right\} \quad (331)$$

$$ABCDE \approx \text{rect. } OP \cdot \frac{1}{2}(AB + BC + CD + DE + EA), \text{ Q.E.D.}$$

$$\text{since } ABCDE = \text{the sum of all the } \Delta, \quad (\text{Ax. 9})$$

$$\text{and } OP = OP_2 = OP_3, \text{ etc.} \quad (162)$$

SCHOLIUM. In the case of the triangle, the most frequent application of this theorem may be stated as follows:

The area of a triangle is equal to the product of half its perimeter and the radius of the inscribed circle.

EXERCISE 463. In diagram for Prop. VI., show that

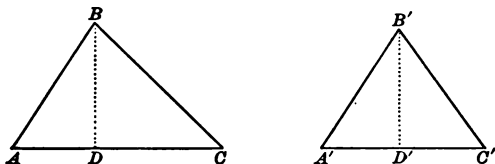
$$\text{rect. } AB \cdot (AE - EB) \approx \overline{AE}^2 - \overline{EB}^2.$$

464. In the same diagram, show that $\text{rect. } AE \cdot (AB + EB) \approx \overline{AB}^2 - \overline{EB}^2$.

465. In the same diagram, if $\angle B = 62^\circ 54' 23''$, what must $\angle D$ be in order that A , B , C , and D may be concyclic points?

PROPOSITION VIII. THEOREM.

341. *Triangles that have an angle of the one equal to an angle of the other, are to each other as the rectangles contained by the sides including those angles.*



Given: Two triangles, ABC , $A'B'C'$, having angle A equal to angle A' ;

To Prove: Triangle ABC : triangle $A'B'C'$ = rect. $AB \cdot AC$: rect. $A'B' \cdot A'C'$.

Draw BD , $B'D'$, \perp to AC , $A'C'$, resp.

Since $\angle A = \angle A'$ (Hyp.), and $\angle D$, $\angle D'$, are rt. \angle s,

$$\triangle BDA \text{ is similar to } \triangle B'D'A'; \quad (286)$$

$$\therefore AB : BD = A'B' : B'D'.$$

$$\left. \begin{array}{l} \text{Since rect. } AB \cdot AC : \text{rect. } BD \cdot AC = AB : BD, \\ \text{and rect. } A'B' \cdot A'C' : \text{rect. } B'D' \cdot A'C' = A'B' : B'D', \end{array} \right\} \quad (317)$$

$$\text{rect. } AB \cdot AC : A'B' \cdot A'C' = \text{rect. } BD \cdot AC : \text{rect. } B'D' \cdot A'C'. \quad (232''')$$

$$\text{But } \triangle ABC : \triangle A'B'C' = \text{rect. } BD \cdot AC : \text{rect. } B'D' \cdot A'C'; \quad (334)$$

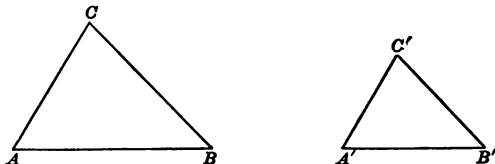
$$\therefore \triangle ABC : \triangle A'B'C' = \text{rect. } AB \cdot AC : \text{rect. } A'B' \cdot A'C'.$$

Q.E.D. (232''')

SCHOLIUM. The theorem may be expressed also under the form: *The areas of triangles that have an angle of the one equal to an angle of the other, are as the products of the sides about those angles.*

PROPOSITION IX. THEOREM.

342. *Similar triangles are to each other as the squares of their homologous sides.*



Given: Similar triangles ABC , $A'B'C'$, having $AB : AC = A'B' : A'C'$;

To Prove: Triangle ABC : triangle $A'B'C' = \overline{AB}^2 : \overline{A'B'}^2$, etc.

$$\left. \begin{array}{l} \text{Since } \overline{AB}^2 : \text{rect. } AB \cdot AC = AB : AC, \\ \text{and } \overline{A'B'}^2 : \text{rect. } A'B' \cdot A'C' = A'B' : A'C', \end{array} \right\} \quad (317)$$

$$\overline{AB}^2 : \overline{A'B'}^2 = \text{rect. } AB \cdot AC : \text{rect. } A'B' \cdot A'C', \quad (249, 244)$$

(since $AB : AC = A'B' : A'C'$.) (Hyp.)

But $\triangle ABC : \triangle A'B'C' = \text{rect. } AB \cdot AC : \text{rect. } A'B' \cdot A'C'$, (341)

(since $\angle A = \angle A'$;))

$$\therefore \triangle ABC : \triangle A'B'C' = \overline{AB}^2 : \overline{A'B'}^2. \quad \text{Q.E.D.} \quad (232''')$$

343. *COR. Similar triangles are to each other as the squares of any homologous lines.*

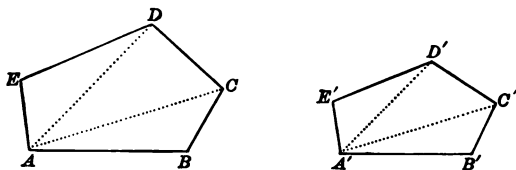
EXERCISE 466. Show that the square of the sum of two lines diminished by the square of their difference is equivalent to four times the rectangle contained by the two lines.

467. In the diagram for Prop. IX., if CD , $C'D'$, be drawn, making the same angle with AB , $A'B'$, resp., $\triangle ABC : \triangle A'B'C' = \overline{CD}^2 : \overline{C'D'}^2$.

468. In the same diagram, what should be the ratio of AB to $A'B'$ so that $\triangle ABC$ may have twice the area of $\triangle A'B'C'$?

PROPOSITION X. THEOREM.

344. Any two similar polygons are to each other as the squares of their homologous sides.



Given: Similar polygons $ABCDE$ or P , and $A'B'C'D'E'$ or P' , and AB homologous to $A'B'$;

To Prove: $P : P' = \overline{AB}^2 : \overline{A'B'}^2$.

Draw diagonals from A and A' , dividing the polygons into the same number of homologous triangles. (295)

Since any homologous pair of these triangles, as ABC , $A'B'C'$, or ACD , $A'C'D'$, are as the squares of any homologous sides, (342)

$$\triangle ABC : \triangle A'B'C' = \triangle ACD : \triangle A'C'D' = \overline{AB}^2 : \overline{A'B'}^2, \text{ etc.};$$

$$\therefore \triangle ABC + \triangle ACD + \triangle ADE : \triangle A'B'C' + \triangle A'C'D' + \triangle A'D'E' = \overline{AB}^2 : \overline{A'B'}^2; \quad (251)$$

$$\therefore P : P' = \overline{AB}^2 : \overline{A'B'}^2. \quad \text{Q.E.D.}$$

345. COR. 1. Similar polygons are to each other as the squares of any homologous lines.

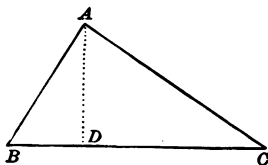
346. COR. 2. The homologous sides of any two similar polygons are as the square roots of the areas of those polygons.

EXERCISE 469. In the diagram for Prop. X., if $AB : A'B' = m : n$, what will be the ratio of the square described on AD to that described on $A'D'$?

470. In the same diagram, if $P : P' = m : n$, what is the ratio of any two homologous lines in the figures, as AC and $A'C'$?

PROPOSITION XI. THEOREM.

347. *In a right triangle, the square of the hypotenuse is equivalent to the sum of the squares of the arms.*



Given: A right triangle ABC , having the right angle at A ;

To Prove: \overline{BC}^2 is equivalent to $\overline{AB}^2 + \overline{AC}^2$.

Draw $AD \perp$ to BC .

$$\text{since } BC : AB = AB : BD, \quad (297)$$

$$\text{rect. } BC \cdot BD \approx \overline{AB}^2; \quad (321)$$

$$\text{since } BC : AC = AC : DC, \quad (297)$$

$$\text{rect. } BC \cdot DC \approx \overline{AC}^2; \quad (321)$$

$$\therefore \text{rect. } BC \cdot BD + \text{rect. } BC \cdot DC \approx \overline{AB}^2 + \overline{AC}^2; \quad (\text{Ax. 2})$$

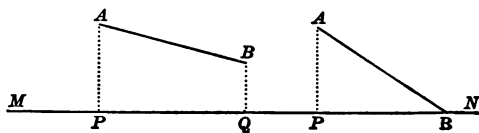
$$\text{i.e., rect. } BC \cdot (BD + DC) \approx \overline{AB}^2 + \overline{AC}^2; \quad (335)$$

$$\text{i.e., } \overline{BC}^2 \approx \overline{AB}^2 + \overline{AC}^2. \quad \text{Q.E.D.}$$

348. COR. *If any similar polygons are constructed on the sides of a right triangle, that on the hypotenuse is equivalent to the sum of those on the arms.*

349. DEFINITIONS. The *projection* of a point upon an indefinite straight line is the foot of the perpendicular drawn from the point to the line.

Thus P is the projection of the point A upon MN .



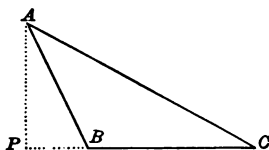
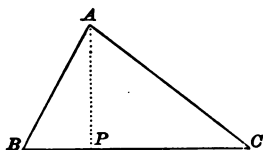
The *projection of a finite straight line* upon an indefinite straight line is the intercept between the projections of the extremities of the line.

Thus PQ is the projection of AB upon MN . If B , one of the extremities of AB , is in MN , then B coincides with its own projection, and PB is the projection of AB .



PROPOSITION XII. THEOREM.

350. *In any triangle, the square of a side subtending an acute angle is less than the sum of the squares of the other sides by twice the rectangle contained by either of these sides and the projection upon it of the other side.*



Given: C , an acute angle of triangle ABC , and PC , the projection on BC of AC ;

To Prove: \overline{AB}^2 is equivalent to $\overline{BC}^2 + \overline{AC}^2 - 2 \text{ rect. } BC \cdot PC$.

According as P is in BC or in BC produced,

$PB = BC - PC$, or $= PC - BC$. In either case,

$$\overline{PB}^2 = \overline{BC}^2 + \overline{PC}^2 - 2 \text{ rect. } BC \cdot PC; \quad (336)$$

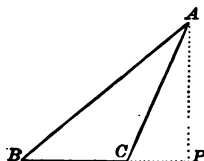
$$\therefore \overline{PB}^2 + \overline{AP}^2 = \overline{BC}^2 + \overline{PC}^2 + \overline{AP}^2 - 2 \text{ rect. } BC \cdot PC. \quad (\text{Ax. 2})$$

$$\text{But } \overline{PB}^2 + \overline{AP}^2 = \overline{AB}^2, \text{ and } \overline{PC}^2 + \overline{AP}^2 = \overline{AC}^2; \quad (347)$$

$$\therefore \overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2 \text{ rect. } BC \cdot PC. \quad \text{Q.E.D.}$$

PROPOSITION XIII. THEOREM.

351. In an obtuse triangle, the square of the side subtending the obtuse angle is greater than the sum of the squares of the other sides by twice the rectangle of either of these sides and the projection upon it of the other side.



Given: C , the obtuse angle of triangle ABC , and PC , the projection on BC of AC ;

To Prove: \overline{AB}^2 is equivalent to $\overline{BC}^2 + \overline{AC}^2 + 2 \text{ rect. } BC \cdot PC$.

Since $PB = BC + PC$,

$$\overline{PB}^2 \approx \overline{BC}^2 + \overline{PC}^2 + 2 \text{ rect. } BC \cdot PC; \quad (336)$$

$$\therefore \overline{PB}^2 + \overline{AP}^2 \approx \overline{BC}^2 + \overline{PC}^2 + \overline{AP}^2 + 2 \text{ rect. } BC \cdot PC. \quad (\text{Ax. 2})$$

$$\text{But } \overline{PB}^2 + \overline{AP}^2 \approx \overline{AB}^2, \text{ and } \overline{PC}^2 + \overline{AP}^2 = \overline{AC}^2; \quad (347)$$

$$\therefore \overline{AB}^2 \approx \overline{BC}^2 + \overline{AC}^2 + 2 \text{ rect. } BC \cdot PC. \quad \text{Q.E.D.}$$

352. COR. If the square of one side of a triangle is equivalent to the sum or the difference of the squares of the other two sides, in either case the two lesser sides are at right angles to each other (347, 350, 351).

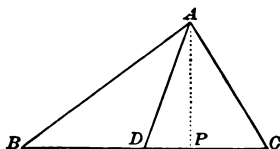
EXERCISE 471. Show that the difference of the square of a line and the square of its projection on another line, is equivalent to the square of the difference of the perpendiculars that intercept the projection.

472. In the diagram for Prop. VI., show that $\overline{AB}^2 + \overline{BC}^2 \approx \overline{AC}^2 + 2 \text{ rect. } AB \cdot EB$, and $\overline{CD}^2 + \overline{AD}^2 = \overline{AC}^2 - 2 \text{ rect. } CD \cdot (AE - CD)$.

473. Show, by means of the foregoing exercise, that $\overline{AB}^2 + \overline{BC}^2 - (\overline{CD}^2 + \overline{AD}^2) \approx 2(\text{rect. } AB(AB - AE) + \text{rect. } CD(AE - CD))$.

PROPOSITION XIV. THEOREM.

353. *The sum of the squares of any two sides of a triangle is equivalent to twice the square of the median to the third side plus twice the square of half this side.*



Given: In triangle ABC , AD the median drawn to BC ;

To Prove: $\overline{AB}^2 + \overline{AC}^2$ is equivalent to $2\overline{AD}^2 + 2\overline{BD}^2$.

Draw $AP \perp$ to BC .

1°. If $AB = AC$, then AP coincides with AD , (74)

$$\overline{AB}^2 \approx \overline{BD}^2 + \overline{AD}^2, \text{ and } \overline{AC}^2 \approx \overline{DC}^2 + \overline{AD}^2; \quad (347)$$

$$\therefore \overline{AB}^2 + \overline{AC}^2 \approx 2\overline{BD}^2 + 2\overline{AD}^2, \text{ Q.E.D. (Ax. 2)}$$

(since $DC = BD$.)

2°. If $AB > AC$, $\angle ADB$ is obtuse and $\angle ADC$, acute;

$$\therefore \overline{AB}^2 \approx \overline{AD}^2 + \overline{BD}^2 + 2BD \cdot DP, \quad (351)$$

$$\text{and } \overline{AC}^2 \approx \overline{AD}^2 + \overline{DC}^2 - 2DC \cdot DP; \quad (350)$$

$$\therefore \overline{AB}^2 + \overline{AC}^2 \approx 2\overline{AD}^2 + 2\overline{BD}^2. \text{ Q.E.D. (Ax. 2)}$$

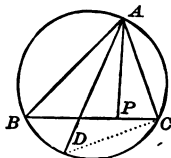
EXERCISE 474. In the diagram for Prop. XII., left-hand figure, if BP' be drawn perpendicular to AC , show that $\text{rect. } AC \cdot P'C \approx \text{rect. } BC \cdot PC$.

475. In the same diagram, if $\overline{AB}^2 \approx \overline{AC}^2 + 3PC^2$, in what way does P divide BC ?

476. From the diagram for Exercise 474 deduce a proof of the theorem: The altitudes of a triangle are inversely proportional to the sides to which they are drawn.

PROPOSITION XV. THEOREM.

354. *The rectangle of any two sides of a triangle is equivalent to the rectangle contained by the altitude upon the third side and the diameter of the circumscribed circle.*



Given : Triangle ABC inscribed in circle $ABDC$, AD a diameter, and AP perpendicular to BC ;

To Prove : Rect. $AB \cdot AC$ is equivalent to rect. $AP \cdot AD$.

Join DC .

Since ACD is a semicircle, (Hyp.)

$$\text{rt. } \angle ACD = \text{rt. } \angle APB; \quad (267)$$

$$\text{also } \angle B = \angle D; \quad (266)$$

$$\therefore \triangle APB \text{ is similar to } \triangle ACD; \quad (288)$$

$$\therefore AB : AP = AD : AC; \quad (284)$$

$$\therefore \text{rect. } AB \cdot AC \approx \text{rect. } AP \cdot AD. \text{ Q.E.D. } (321)$$

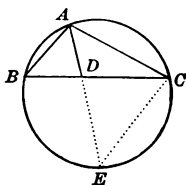
EXERCISE 477. If AC is a diagonal of a parallelogram $ABCD$, having $\angle A$ equal to an angle of an equilateral triangle, show that $AC^2 \approx AB^2 + BC^2 + \text{rect. } AB \cdot BC$.

478. If AC is a diagonal of a parallelogram $ABCD$, having $\angle A$ equal to twice an angle of an equilateral triangle, show that $AC^2 \approx AB^2 + BC^2 - \text{rect. } AB \cdot BC$.

479. The square of the base of an isosceles triangle is equivalent to twice the rectangle contained by either of the arms and the projection of the base upon that side.

PROPOSITION XVI. THEOREM.

355. *The rectangle of any two sides of a triangle is equivalent to the rectangle of the segments made on the third side by the bisector of the opposite angle, plus the square of the bisector.*



Given: In triangle ABC , AD the bisector of angle A , cutting BC in D ;

To Prove: $\text{rect. } AB \cdot AC$ is equivalent to $\text{rect. } BD \cdot DC + \overline{AD}^2$.

Describe $\odot ABEC$ about $\triangle ABC$; (185)

produce AD to meet circumference in E , and join EC .

Since $\angle BAD = \angle CAE$, (Hyp.)

and $\angle B = \angle E$, (266)

$\triangle ABD$ is similar to $\triangle AEC$; (287)

$\therefore AB : AD = AE : AC$; (284)

$\therefore \text{rect. } AB \cdot AC \approx \text{rect. } AD \cdot AE$; (321)

i.e., $\text{rect. } AB \cdot AC \approx \text{rect. } AD \cdot (AD + DE)$;

i.e., $\text{rect. } AB \cdot AC \approx \overline{AD}^2 + \text{rect. } AD \cdot DE$; (335)

i.e., $\text{rect. } AB \cdot AC \approx \text{rect. } BD \cdot DC + \overline{AD}^2$. Q.E.D. (301)

EXERCISE 480. The sum of the squares of the sides of any parallelogram is equivalent to the sum of the squares of the diagonals.

481. A median divides a triangle into two equivalent triangles.

482. Three times the sum of the squares of the sides of a triangle is equivalent to four times the sum of the squares of the medians.

PROPOSITION XVII. THEOREM.

356. *If two chords intersect, the rectangle of the segments of the one is equivalent to the rectangle of the segments of the other.*

(See diagram for Prop. XXI., Book IV.)

$$\text{Since } OA : OD = OC : OB, \quad (301)$$

$$\text{rect. } OA \cdot OB \approx \text{rect. } OC \cdot OD. \quad \text{Q.E.D.} \quad (321)$$



PROPOSITION XVIII. THEOREM.

357. *If from the same point a tangent and a secant be drawn to a circle, the square of the tangent is equivalent to the rectangle of the secant and its external segment.*

(See diagram for Prop. XXIII., Book IV.)

$$\text{Since } OA : OC = OC : OB, \quad (303)$$

$$\overline{OC}^2 \approx \text{rect. } OA \cdot OB. \quad \text{Q.E.D.} \quad (322)$$

SCHOLIUM. The two foregoing propositions enunciate the last three of Book IV. from a different point of view. In these and similar theorems, we may substitute *product* for *rectangle* when regard is had to the numerical measures of the lines concerned.

EXERCISE 483. In any triangle ABC , if the altitudes BD , CE , be drawn to AC , AB , respectively, show that $\overline{BC}^2 \approx \text{rect. } AB \cdot BE + \text{rect. } AC \cdot CD$.

484. If ABC is a scalene triangle,

$$\overline{AB}^2 + \overline{AC}^2 + \overline{BC}^2 > AB \cdot AC + AB \cdot BC + AC \cdot BC.$$

485. The square of the median to the hypotenuse of a right triangle is equivalent to one fourth of the square of the hypotenuse.

EXERCISES.

THEOREMS.

486. A parallelogram is divided by its diagonals into four equivalent triangles.

487. If two triangles have two sides of the one severally equal to two sides of the other, and the included angles supplementary, the triangles are equivalent.

488. If any point within a parallelogram be joined with the vertices, the sums of the opposite pairs of triangles are equivalent.

489. If through any point in a diagonal of a parallelogram parallels to the sides be drawn, of the four parallelograms thus formed, the two through which the diagonal does not pass are equivalent.

490. A line joining the mid points of its bases bisects a trapezoid.

491. If the mid points of the sides of a quadrilateral be joined in order, a parallelogram is formed equivalent to one half the quadrilateral.

492. The lines joining the mid point of a diagonal of a quadrilateral with the opposite vertices, cut off one half the quadrilateral.

493. The sum of the squares of the sides of any quadrilateral is equivalent to the sum of the squares of the diagonals, and four times the square of the line that joins their mid points.

494. If from any point P in the production of AC , a diagonal of a parallelogram $ABCD$, lines PB , PD , be drawn, the triangles PBC , PDC , will be equivalent.

495. If from a point P without a parallelogram $ABCD$, lines PA , PB , PC , PD , be drawn, then $\triangle PAB - \triangle PCD$ is equivalent to one half the parallelogram.

496. Of the four triangles formed by drawing the diagonals of a trapezoid, (1) those having as bases the nonparallel sides are equivalent; (2) those having as bases the parallel sides are as the squares of those sides.

497. If two triangles have a common angle and equal areas, the sides containing the common angle are inversely proportional.

498. If through its vertices lines be drawn parallel to the diagonals of any quadrilateral, the figure formed will be a parallelogram of twice the area of the quadrilateral.

499. In the diameter of a circle two points are taken equally distant from the center; if through one of these any chord be drawn, and its extremities joined with the other point, the sum of the squares of the triangle formed is constant.

500. The sum of the squares of the four segments of any two chords that intersect at right angles is constant.

501. The rectangle of the segments of chords intersecting at a given distance from the center is constant.

502. The square inscribed in a semicircle is to that inscribed in the circle as 2 is to 5.

503. If one side of a triangle is lengthened, and another shortened by the same length, the line joining the points of section is divided by the base in the inverse proportion of the sides.

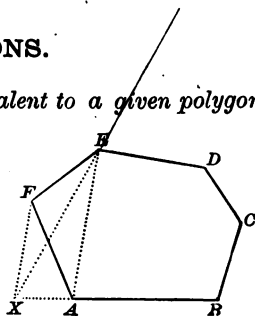
504. If from the same point a secant and two tangents be drawn, the secant will be divided harmonically by the circumference and the chord joining the points of contact.

CONSTRUCTIONS.

358. *To construct a triangle equivalent to a given polygon $AB \dots F$.*

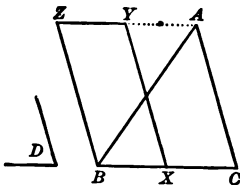
Join the extremities of any two adjacent sides, AF , FE , by AE ; and through F draw FX parallel to AE , to meet BA produced in X . Join EX . We have now a polygon $BCDEX$, having one side fewer than the given polygon, and equivalent to it, since $\triangle XEA \approx \triangle FEA$ (332).

By now joining BE , and proceeding as before, we obtain an equivalent polygon having one side fewer than $BCDEX$, and two fewer than the given polygon. Continuing the process, we evidently must at last obtain a triangle equivalent to the given polygon.



359. To construct a parallelogram equivalent to a given triangle ABC , and having an angle equal to a given angle D .

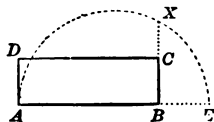
Through A draw AZ parallel to BC , and through X , the mid point of BC , draw XY , making $\angle BXY = \angle D$, and meeting AZ in Y . Through B draw BZ parallel to XY to meet AZ in Z . Then XZ , which is equivalent to $\triangle ABC$ (331), is the parallelogram required.



SCHOLIUM. If the given angle D is a right angle, then XZ will be a rectangle.

360. To construct a square equivalent to a given rectangle AC .

Produce AB to E , so that $BE = BC$, and upon AE as diameter describe a semicircle AXE . Produce BC to meet the circumference in X . Then BX is a side of the required square.



$$\text{For since } AB : BX = BX : BE, \quad (298)$$

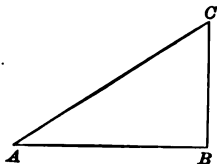
$$\overline{BX}^2 \approx AB \cdot BE \approx AB \cdot BC. \quad (322)$$

SCHOLIUM. By means of the three foregoing constructions, we can construct a square equivalent to any given polygon by first transforming the given polygon into a triangle, then the triangle into a rectangle, and finally the rectangle into a square.

361. To construct a square equivalent to the sum of two given squares.

Draw $BC \perp$ to AB , and make AB , BC , respectively equal to the sides of the given squares. Join AC . Then

$$\overline{AC}^2 \approx \overline{AB}^2 + \overline{BC}^2.$$

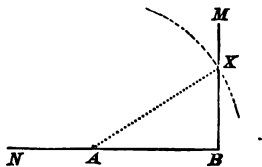


SCHOLIUM. It is obvious that, by a continuation of the process, we can obtain the side of a square equivalent to the sum of any number of given squares.

362. *To construct a square equivalent to the difference of two given squares.*

Draw $BM \perp$ to BN , and on BN lay off $BA =$ the side of the lesser square. From A as center, with radius $AX =$ the side of the greater square, describe an arc cutting BM in X . Then BX is a side of the required square. For

$$\overline{BX}^2 = \overline{AX}^2 - \overline{AB}^2. \quad (347)$$



363. *To construct a square that shall be any given part of a given square.*

That is, L being the side of the given square, we have to find a line X , such that

$$X : L = \sqrt{m} : \sqrt{n}, \text{ or } X^2 = \frac{m}{n} L^2.$$

Employing the construction of Art. 362, we take $BA = \frac{m-n}{2n} L$ and $AX = \frac{m+n}{2n} L$; that is, we divide L into $2n$ equal parts (207), then take $m-n$ and $m+n$ such parts.

$$\overline{BX}^2 \text{ or } X^2 = \left(\frac{m+n}{2n}\right)^2 L^2 - \left(\frac{m-n}{2n}\right)^2 L^2 = \frac{m}{n} L^2.$$

If, for example, we wish to find the side of a square that shall be equivalent to $\frac{11}{6}$ of a given square whose side is a line L , we divide L into 12 equal parts (207), take $BA = (11-6)$, or 5, of these parts, and $AX = (11+6)$, or 17, of these parts. Then

$$\overline{BX}^2 = \left(\frac{17}{12}\right)^2 L^2 - \left(\frac{5}{12}\right)^2 L^2 = \frac{11}{6} L^2, \text{ or } X : L = \sqrt{11} : \sqrt{6}.$$

If, again, we wish to find a line whose numerical measure shall be $\sqrt{13}$, supposing L to be the linear unit, we

take $BA = \frac{13-1}{2}L$, or $6L$, and $AX = \frac{13+1}{2}L$, or $7L$. Then, since $\overline{BX}^2 = 7^2L^2 - 6^2L^2 = 13L^2$, we have $BX = \sqrt{13}L$; i.e., the numerical measure of BX is $\sqrt{13}$.

364. To construct a polygon P similar to a given polygon Q , so that we shall have

$$P : Q = m : n.$$

Let L be any side of Q ; then by Art. 363 find a line x such that

$$x : L = \sqrt{m} : \sqrt{n}.$$

Upon x construct (310) a polygon P similar to the given polygon Q . Then P is the polygon required; since (344) $P : Q = (\sqrt{m})^2 : (\sqrt{n})^2 = m : n$.

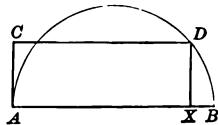
365. To construct a polygon similar to each of two given polygons, and equivalent to their sum or difference.

We obtain an homologous side of the required polygon by proceeding with two homologous sides of the given polygons as we did with the sides of the given squares in Art. 361, when we wish to find a sum; and as we did with the sides of the given squares in Art. 362, when we wish to find a difference. Upon the base thus obtained we then construct (310) a polygon similar to the given polygons.

366. To construct a rectangle equivalent to a given square, and having the sum of two adjacent sides equal to a given line AB .

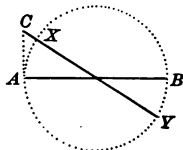
Upon AB as diameter, describe a semicircumference ADB . At A draw $AC \perp$ to AB , and equal to the side of the given square. Through C draw $CD \parallel$ to AB to meet ADB in D . From D draw $DX \perp$ to AB to meet AB in X . Then AX and BX are respectively the base and altitude of the required rectangle. For

$$\text{rect. } AX \cdot BX \doteq DX^2 \doteq AC^2. \quad (298)$$



367. To construct a rectangle equivalent to a given square, and having the difference of two adjacent sides equal to a given line AB .

Upon AB as diameter, describe a circumference $AXBY$. At A draw $AC \perp$ to AB , and equal to the side of the given square. From C draw the secant CXY through the center to meet the circumference in X and Y . Then CX , CY are, respectively, the base and altitude of the required rectangle. For



$$\text{rect. } CY \cdot CX = AC^2. \quad (357)$$

368. To construct a polygon similar to a given polygon P , and equivalent to another given polygon Q .

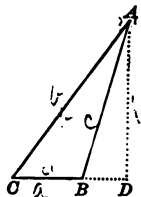
Find (360, Scholium) x and y , the sides of squares equivalent to P and Q respectively. Then (305) find a fourth proportional z , to x , y , and L , the base of P . The polygon constructed on z as base and similar to P (310) will be the required polygon. *Proof by the student.*

NUMERICAL APPLICATIONS.

1°. To compute the altitude of any triangle in terms of its sides.

In the $\triangle ABC$, let a , b , c , be the numerical measure of the sides opposite A , B , C , respectively, and h that of the altitude from A .

Of the $\angle B$ and C , one at least, say C , must be acute;



$$\therefore h^2 = b^2 - \overline{DC}^2; \quad (347)$$

$$\text{and } c^2 = a^2 + b^2 - 2a \cdot CD; \quad (350)$$

$$\therefore CD = \frac{a^2 + b^2 - c^2}{2a};$$

$$\begin{aligned}
 \therefore h^2 &= b^2 - \left(\frac{a^2 + b^2 - c^2}{2a} \right)^2 = \frac{4a^2b^2 - (a^2 + b^2 - c^2)^2}{4a^2} \\
 &= \frac{(2ab + a^2 + b^2 - c^2)(2ab - a^2 - b^2 + c^2)}{4a^2} \\
 &= \frac{\{(a+b)^2 - c^2\}\{c^2 - (a-b)^2\}}{4a^2} \\
 &= \frac{(a+b+c)(a+b-c)(c+a-b)(c-a+b)}{4a^2}.
 \end{aligned}$$

Let $a + b + c = 2s$; i.e., let s denote half the perimeter;

$$\text{then } a + b - c = 2(s - c);$$

$$c + a - b = 2(s - b);$$

$$c - a + b = 2(s - a).$$

Hence, simplifying and extracting the square root,

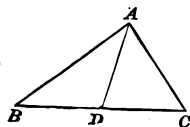
$$h = \frac{2}{a} \sqrt{s(s-a)(s-b)(s-c)}.$$

2°. To compute the medians of a triangle in terms of the sides.

In $\triangle ABC$, denoting the values of the sides as in 1°, and the median from A by m ,

$$\text{since } b^2 + c^2 = 2m^2 + 2\left(\frac{1}{2}a\right)^2, \quad (353)$$

$$m = \frac{1}{2} \sqrt{2(b^2 + c^2) - a^2}.$$



3°. To compute the bisectors of a triangle in terms of the sides.

In $\triangle ABC$, denoting the values of the sides as in 1°, and the bisector from A , by d (see diagram for Prop. XVI.);

$$\text{since } bc = BD \times DC + d^2, \quad (355)$$

$$d^2 = bc - BD \times DC. \quad (1^\circ)$$

$$\text{But } \frac{BD}{c} = \frac{DC}{b} = \frac{BD + DC}{b + c} = \frac{a}{b + c}; \quad (278)$$

$$\therefore BD = \frac{ac}{b + c} \text{ and } DC = \frac{ab}{b + c}.$$

Substituting these values in 1°, and simplifying,

$$d = \frac{2}{b + c} \sqrt{bcs(s - a)}.$$

4°. *To compute the radius of the circumscribed circle in terms of the sides of a triangle ABC.*

In $\triangle ABC$, denoting the values of the sides as in 1°, and the radius by R (see diagram for Prop. XV.);

$$\text{since } bc = 2R \times AP, \quad (354)$$

$$\text{and } AP = \frac{2}{a} \sqrt{s(s - a)(s - b)(s - c)}, \quad (1^\circ)$$

$$R = \frac{abc}{4 \sqrt{s(s - a)(s - b)(s - c)}}.$$

5°. *To compute the area of a triangle in terms of its sides.*

Denoting the values of the sides as in 1°, that of the altitude by h , and of the area by s ,

$$\text{since } s = \frac{1}{2} ah \text{ (331), and } h = \frac{2}{a} \sqrt{s(s - a)(s - b)(s - c)}, \quad (1^\circ)$$

$$s = \sqrt{s(s - a)(s - b)(s - c)}.$$

SCHOLIUM. If the triangle is equilateral, i.e., if $a = b = c$, the formula reduces to $s = \frac{a^2}{4} \sqrt{3}$.

6°. *To compute the area of a triangle in terms of the sides and the radius of the circumscribing circle.*

Denoting the values of the sides, altitude, and area as in 5°,

$$\text{since } bc = 2 R \cdot h, \quad (354)$$

$$\therefore abc = 2 R \cdot a \cdot h = 4 R \cdot S;$$

$$\therefore S = \frac{abc}{4 R}$$

EXERCISES.

QUESTIONS.

505. How many different altitudes can each of the following figures have: An equilateral triangle? An isosceles triangle? A scalene triangle? A square? A rectangle? A trapezoid? A trapezium?

506. A side of an equilateral triangle is 6.* What is its altitude?

507. An arm and the base of an isosceles triangle are 18 and 10 respectively. What are its altitudes?

508. The area of a triangle is 180; its sides are 30, 60, and 40, respectively. What are its altitudes?

509. The area of a triangle is 252; its altitudes are 8, 12, and 14, respectively. What are its sides?

510. The sides of a rectangle are 65 and 32 respectively. What are its area, perimeter, and diagonal?

511. The altitude and base of a triangle being 23 and 10 respectively, what is its area?

512. The area of a triangle is 221 sq. ft.; its base is $5\frac{2}{3}$ yds. What is its altitude in inches?

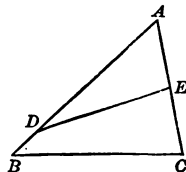
513. The bases of two parallelograms are 15 and 16 respectively; their altitudes are 8 and 10 respectively. What is the ratio of their areas?

514. Two triangles of equal areas have their bases 26 in. and 3 ft. respectively. What is the ratio of their altitudes?

*Remember that the given abstract numbers are numerical measures.

515. The bases of a trapezoid are 23 in. and 17 in. respectively, its altitude being $2\frac{1}{2}$ ft. What is its area?

516. In $\triangle ABC$, $AB = 42$, $AC = 34$. If DE cut off $AD = 30$ and $AE = 15$, what is the ratio of $\triangle ABC$ to $\triangle ADE$?



517. What should be the length of a ladder such that, having its foot 15 ft. from the wall, it may reach a window 20 ft. from the ground?

518. Two chords intersect so that the segments of one are 12 and 7 respectively. If a segment of the other is 10, what is its second segment?

519. What are the altitudes of a triangle whose sides are 12, 15, 9, respectively?

520. What are the medians of the same triangle?

521. What are the bisectors of the angles of the same triangle?

522. What is the radius of the circle circumscribing the same triangle?

523. What is the area of the same triangle?

524. What is the area of a triangle whose sides are 6, 5, 5, respectively?

525. What is the area of an equilateral triangle whose side is 3?

526. What is the area of an equilateral triangle whose altitude is 11?

527. The sides of a right triangle are 25, 24, 7. What are its medians and its altitude upon the hypotenuse?

528. From the same point a tangent and a secant being drawn, if the secant and its external segment are as 27 to 3, what is the length of the tangent?

529. If from the point just referred to a second secant be drawn, so that its external segment is 8, what will be the length of the secant?

530. Two secants drawn from the same point have external segments of 5 and 3 respectively. If the first secant is 27, what are the internal segments?

PROBLEMS.

531. Construct an isosceles triangle on the same base as a given triangle, and equivalent to it.

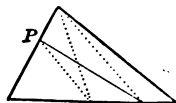
532. Construct a right isosceles triangle equivalent to a given square.

533. Construct a parallelogram having a given angle upon the same base as a given square, and equivalent to it.

534. Divide a given line into two segments such that their squares shall be as 7 is to 5.

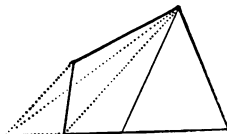
535. Bisect a given parallelogram (1) by a line passing through a given point; (2) by a perpendicular to a side; (3) by a line parallel to a given line.

536. Bisect a given triangle by a line drawn through a given point P in one of the sides.



537. Cut off one n th of a triangle by a line drawn through a given point in one of the sides.

538. Bisect a quadrilateral by a line drawn through one of the vertices.



539. Cut off from a quadrilateral one n th part by a line drawn through one of the vertices.

540. Bisect a triangle by a line parallel to the base.

541. Bisect a triangle by a line perpendicular to the base.

542. Find a point within a triangle such that lines joining the point with the vertices shall divide the triangle into three equivalent triangles.

543. Bisect a trapezoid by a line drawn parallel to the bases.

544. Bisect a trapezoid by a line drawn through a given point in one of the bases.

545. Construct a triangle equivalent to a given triangle, and having one side equal to a given line.

546. Construct a right triangle equivalent to a given triangle, and having one arm of a given length.

547. Construct a right triangle equivalent to a given triangle, and having its hypotenuse of a given length.

548. Construct an isosceles triangle equivalent to a given triangle, and having its arms of a given length.

549. Construct an isosceles triangle equivalent to a given triangle, and having its base of a given length.

550. Construct an equilateral triangle equivalent to a given triangle.

551. Construct an equilateral triangle equivalent to a given square.

552. Construct a triangle similar to each of two given similar triangles, and equivalent to their sum.

553. Construct a triangle similar to each of two given similar triangles, and equivalent to their difference.

554. Construct a square that shall be to a given triangle as 5 is to 3.

555. Construct an equilateral triangle that shall be to a given square as 7 is to 5.

BOOK VI.

REGULAR PLANE FIGURES.



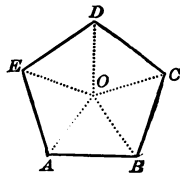
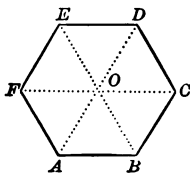
REGULAR POLYGONS.

369. A polygon of five sides is called a *pentagon*; one of six sides is called a *hexagon*; of seven sides, a *heptagon*; of eight sides, an *octagon*; of fifteen sides, a *pentadecagon*; and so on.

370. A *regular polygon* is both equilateral and equiangular.

PROPOSITION I. THEOREM.

371. A *regular polygon* may be divided into as many equal isosceles triangles as the polygon has sides.



Given: A regular polygon $AB \dots F$, or P , having n sides;

To Prove: P may be divided into n equal isosceles triangles.

Bisect $\angle A$ and B by AO , BO . (81)

$\angle A$ and B are each $<$ a st. \angle ;

$\therefore \frac{1}{2} \angle A + \frac{1}{2} \angle B <$ a st. \angle ;

$\therefore AO, BO$, must meet in some point O . (114)

Join O with C, D, E, F ;

then since $\angle OAB = \angle OBA$, (Ax. 7)

$\triangle OAB$ is isosceles. (65)

Since $AB = BC$ (Hyp.), OB is common,

and $\angle OBA = \angle OBC$, (Const.)

$\triangle OBC = \triangle OAB$. (66)

In the same way it may be shown that

$\triangle OCD = \text{isos. } \triangle OBC, \triangle OAF = \text{isos. } \triangle OAB$, etc. Q.E.D.

372. COR. 1. *The bisectors of any two angles of a regular polygon determine by their intersection a point equidistant from the vertices of the polygon.*

For the lines drawn from the vertices to that point are the sides of equal isosceles triangles (371).

373. COR. 2. *The point that is equidistant from the vertices, is also equidistant from the sides, of the polygon (101).*

For it is in the bisectors of all the angles of the polygon.

374. COR. 3. *A circle may be circumscribed about, or inscribed in, any regular polygon, and both circles have the same center.*

For taking the intersection of the bisectors of any two angles as center, the circumference described through one vertex will pass through all (372), and that described tangent to one side will be tangent to all (373).

375. DEFINITION. The *center* of a regular polygon is the common center of the inscribed and circumscribed circles.

376. DEFINITION. The *radius* of a regular polygon is that of the circumscribed circle.

377. DEFINITION. The *apothem* of a regular polygon is the radius of the inscribed circle; i.e., the distance from the center to any side.

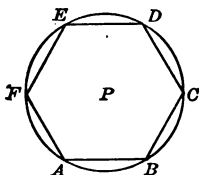
378. DEFINITION. An angle at the center of a regular polygon is the angle formed by the radii drawn to the extremities of any side.

379. COR. 4. If the number of sides is n , each angle at the center $= 2 \text{ st. } \angle \div n$.



PROPOSITION II. THEOREM.

380. An equilateral polygon inscribed in a circle is regular.



Given: An equilateral polygon P inscribed in $\odot ACE$;

To Prove: P is a regular polygon.

Since chd. $AB = \text{chd. } BC = \text{chd. } CD$, etc., (Hyp.)

arc $AB = \text{arc } BC = \text{arc } CD$, etc.; (174)

$\therefore \text{arc } ABC = \text{arc } BCD = \text{arc } CDE$, etc.; (Ax. 2)

$\therefore \angle B = \angle C = \angle D$, etc., (266)

(being inscribed in equal segments;)

$\therefore P$, being equilateral and equiangular, is regular.

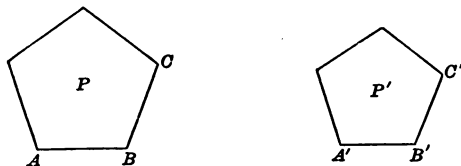
Q.E.D. (370)

381. COR. 1. If a circumference be divided into n equal arcs (n being > 2), the chords subtending these arcs will form a regular polygon of n sides.

382. COR. 2. If the arcs subtended by the sides of a regular polygon of n sides be bisected, the chords drawn to subtend these arcs will form a regular polygon of $2n$ sides.

PROPOSITION III. THEOREM.

383. *Regular polygons of the same number of sides are similar.*



Given: Two regular polygons, ABC or P , and $A'B'C'$ or P' , each of n sides;

To Prove: P is similar to P' .

Since P and P' have each n angles, (Hyp.)

each \angle of P and $P' = \frac{n-2}{n}$ st. \angle resp.; (127)

$\therefore P$ and P' are mutually equiangular.

Since P and P' are each equilateral, (Hyp.)

$$AB : BC = A'B' : B'C';$$

similarly, all the sides about the equal \angle s are proportional;

$\therefore P$ is similar to P' . Q.E.D. (284)

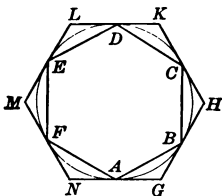
384 COR. *The perimeters of regular polygons of the same number of sides are as their radii, or apothems; and their areas are as the squares of these lines.*

For the radii and apothems are homologous lines; hence the perimeters are proportional to them (296), and the areas are proportional to the squares of those lines (344).

EXERCISE 556. The ratio of an interior angle of a regular polygon of n sides to an interior angle of a regular polygon of double the number of sides, is expressed by $n - 2 : n - 1$.

PROPOSITION IV. THEOREM.

385. *A regular polygon being inscribed in a circle, a similar polygon may be circumscribed about the circle.*



Given: A regular polygon $AB \dots F$, or P , inscribed in $\odot ACE$;

To Prove: A similar polygon can be circumscribed about ACE .

Through each vertex A, B, C , etc., draw tangents NG, GH, HK , etc.

Since $AB = BC = CD$, etc., (Hyp.)

$\angle GAB = \angle GBA = \angle HBC = \angle HCB$, etc., (269)

(being formed by tangents and chords of equal arcs.)

Since each chord is less than a diameter,

each of these equal \angle s is less than a rt. \angle ; (267)

\therefore the tangents through adjacent vertices will meet. (114)

Let them meet in G, H, K, L , etc.

Since isos. $\triangle GAB =$ isos. $\triangle HBC =$ isos. $\triangle KCD$, etc., (63)

$\angle G = \angle H = \angle K =$ etc.; (70)

then $GHK \dots N$ is equiangular.

Also, $GA = GB = HB = HC =$ etc., (70)

$GH = HK = KL =$ etc.; (Ax. 6)

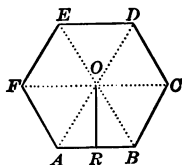
$\therefore GHK \dots N$ is equilateral, with the same no. of sides as P ;

$\therefore GHK \dots N$ is a regular polygon similar to P . Q.E.D. (370)

386. COR. *If a circumference be divided into n equal arcs (n being > 2), the tangents drawn at the points of division will form a regular circumscribed polygon of n sides.*

PROPOSITION V. THEOREM.

387. *Any regular polygon is equivalent to the rectangle of its apothem and half its perimeter.*



Given: OR , the apothem of a regular polygon $AB \dots F$, or P , of n sides;

To Prove: P is equivalent to rectangle $\frac{1}{2} n AB \cdot OR$.

Draw AO , BO .

$$\text{Since } \triangle OAB \approx \frac{1}{2} \text{ rect. } AB \cdot OR, \quad (331)$$

$$\text{and } P \approx n \triangle OAB, \quad (371)$$

$$P \approx \frac{1}{2} n \text{ rect. } AB \cdot OR \approx \text{rect. } \frac{1}{2} n AB \cdot OR. \quad \text{Q.E.D.}$$

SCHOLIUM. The theorem may also be stated thus:

The area of a regular polygon is measured by one half the product of its apothem and perimeter.

EXERCISE 557. In the diagram for Prop. IV., if GR be the perpendicular from G to AB , the number of sides being n , show that GR^2 is $4n$ times the difference of the sums of the squares of the sides of the circumscribed and inscribed polygons.

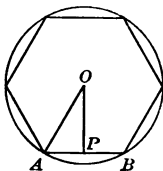
558. In the same diagram, n being the number of sides, each angle formed like GAB by an interior and an exterior side, is $\frac{1}{n}$ of a straight angle.

559. In the diagram for Prop. V., if a parallel to AB be drawn through the mid point of BC , what lines will it bisect?

560. In the same diagram, if the mid points of the radii be joined, what figure will be formed, and what ratio will its area have to that of P ?

PROPOSITION VI. THEOREM.

388. *As the number of sides of a regular inscribed polygon indefinitely increases, the apothem increases towards the radius as its limit.*



Given: AB , a side, and OP , the apothem, of a regular polygon of n sides inscribed in a circle whose radius is OA ;

To Prove: As n increases, OP increases towards OA as limit.

Since OP is \perp to AB , (377)

$$\overline{OP}^2 \approx \overline{OA}^2 - \overline{AP}^2. \quad (347)$$

Now as n increases, AB , and therefore AP , decreases (181), and when n becomes indefinitely great, AP becomes indefinitely small, while OA , the radius, is constant;

$$\therefore \overline{OP}^2 \text{ has for limit } \overline{OA}^2; \quad (235)$$

$$\therefore OP \text{ has for limit } OA. \quad \text{Q.E.D.}$$

EXERCISE 561. In the diagram for Prop. VI., if AO be produced, show that it will pass through a vertex of the polygon.

562. In any polygon of an even number of sides, the lines joining opposite vertices are diameters of the circumscribed circle.

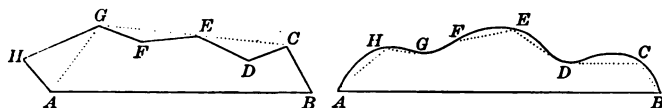
563. In the diagram for Prop. VI., if the inscribed polygon is a regular hexagon, what is the ratio of OP to AP ?

564. In the same diagram, if OB be joined, and PC , PD , be drawn to the mid points of OA , OB , resp., $OPCD$ will be a rhombus, unless $\angle AOB$ is a right angle. In what case will $\angle AOB$ be a right angle?

565. The area of the regular inscribed hexagon is half the area of the circumscribed equilateral triangle.

PROPOSITION VII. THEOREM.

389. *A straight line is the least of all the lines that terminate in two given points.*



Given : A straight line AB , and another line $AHG \dots B$, terminating in A, B ;

To Prove : AB is less than $AHG \dots B$.

Join alternate points AG, GE, EC .

Since $AG < AH + HG$, $GE < GF + FE$, $EC < ED + DC$, (88)

$AGECB < AHGFEDCB$. (Ax. 4)

By continuing the process, as the number of parts is always decreasing, we shall evidently obtain at last a broken line of two parts, which is less than any of the preceding broken lines, and greater than AB .

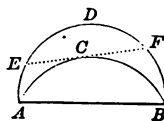
Hence AB is less than any broken line terminating in A, B .

As this reasoning holds true, no matter how numerous or how small the parts of the broken line may be, it holds true in regard to curves also, which are the limits towards which tend the broken lines formed by joining points of the curve taken indefinitely near to each other.

Hence AB is less than any other line terminating in A and B . Q.E.D.

390. COR. 1. *An arc ACB of a circle is less than any enveloping line ADB that terminates in the same points, A and B .*

Join AB . Of all the lines enveloping the area ACB , there must be a least one. Now ADB cannot be the least, for drawing EF tangent to ACB at C , we have $AEFB < ADB$,



since $EF < EDF$. In the same way it can be shown that no other line than ACB can be the least line enveloping the area ACB .

391. COR. 2. *A circumference is greater than the perimeter of any inscribed polygon, and less than that of any circumscribed polygon.*

PROPOSITION VIII. THEOREM.

392. *As the number of their sides indefinitely increases, the perimeters of regular inscribed and circumscribed polygons tend towards the circumference as their common limit, and their areas towards the circle as their common limit.*

Given: The perimeters p, p' , of two regular polygons P, P' , of n sides, circumscribed about, and inscribed in, a circle whose circumference is C and whose area is S ;

To Prove: $\begin{cases} 1^\circ. & p \text{ and } p' \text{ each tend towards } C \text{ as limit.} \\ 2^\circ. & P \text{ and } P' \text{ each tend towards } S \text{ as limit.} \end{cases}$

Let R be the apothem of P and radius of P' , and r the apothem of P' .

$$1^\circ. \quad \text{Since } p:p' = R:r, \quad (384)$$

$$\frac{p-p'}{p} = \frac{R-r}{R}. \quad (247)$$

Now when the number of sides becomes indefinitely great, r approaches its limit R (388), so that $R-r$ becomes indefinitely small. Hence $\frac{R-r}{R}$ and its equal $\frac{p-p'}{p}$ become indefinitely small; that is, $p-p'$, the difference of the perimeters of the polygons, becomes less than any assignable quantity, p being not much greater than C . Now as C

always lies between p and p' (391), its difference from either must be less than $p - p'$; hence

p and p' have each for limit C . Q.E.D. (235)

2°. Since $P : P' = R^2 : r^2$, (384)

$$\frac{P - P'}{P} = \frac{R^2 - r^2}{R^2}. \quad (247)$$

Now when $R^2 - r^2$ becomes indefinitely small, $P - P'$ becomes indefinitely small; and since always

$P > S$, but $P' < S$, (Ax. 8)

S lies always between P and P' ,

(its difference from either being less than $P - P'$;))

$\therefore P$ and P' have each for limit S . Q.E.D.

EXERCISE 566. The apothem of an inscribed equilateral triangle is equal to half the radius of the circle.

567. The apothem of an inscribed regular hexagon is equal to half the side of the inscribed equilateral triangle.

568. Any arc is greater than its chord, but less than the sum of the tangents drawn from a point to its extremities.

569. Every equilateral polygon inscribed in a circle is also equiangular.

570. Every equiangular polygon inscribed in a circle is also equilateral, if the number of sides is odd.

571. Every equilateral polygon circumscribed about a circle is also equiangular, if the number of sides is odd.

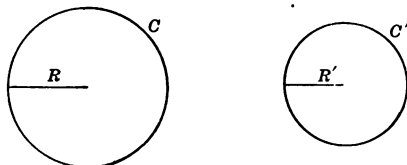
572. Every equiangular polygon circumscribed about a circle is also equilateral.

573. If the alternate vertices of a regular hexagon are joined by straight lines, show that the figure formed is a regular hexagon.

574. What ratio has the latter figure to the former ?

PROPOSITION IX. THEOREM.

393. *Circumferences are to each other as their radii.*



Given: Two circumferences, C , C' , with radii R , R' ,

To Prove: $C : C' = R : R'$.

Conceive regular polygons of n sides to be inscribed in each of the circumferences (381), and by continual doubling (382), the number of sides to become indefinitely great.

Let p , p' , denote the variable perimeters.

$$\text{Since } p : p' = R : R', \quad (384)$$

$$\frac{p}{R} = \frac{p'}{R'}. \quad (244)$$

Now the limits of these equal variables are equal, C being the limit of p and C' of p' (392). Hence

$$\frac{C}{R} = \frac{C'}{R'}. \quad \text{Q.E.D.} \quad (236)$$

394. COR. 1. *Circumferences are to each other as their diameters.*

$$\text{For } C : C' = R : R'; \quad (393)$$

$$\therefore C : C' = 2R : 2R' = D : D'. \quad (253)$$

395. COR. 2. *The ratio of the circumference to the diameter is constant.*

$$\text{For } C : C' = D : D'; \quad (394)$$

$$\therefore C : D = C' : D'.$$

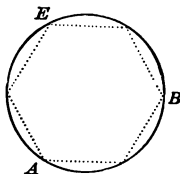
396. SCHOLIUM. The numerical value of this constant ratio,—that is, the number showing how many times a circumference contains its diameter, is denoted by the Greek letter π . It is an incommensurable number, but its value, as will presently be shown, can be obtained to any required degree of precision. Since $\frac{C}{D} = \frac{C}{2R} = \pi$, we have the important relations,

$$C = \pi \cdot D = 2\pi R; \quad D = \frac{C}{\pi}, \quad R = \frac{C}{2\pi}.$$



PROPOSITION X. THEOREM.

397. *A circle is equivalent to one half the rectangle contained by its radius and circumference.*



Given : C , the circumference, and R , the radius, of a circle ABE ;

To Prove : Circle ABE is equivalent to $\frac{1}{2}$ rectangle $R \cdot C$.

Let r denote the apothem, and p the perimeter, of a regular polygon P inscribed in ABE .

$$\text{Then } P \approx \frac{1}{2} \text{ rect. } r \cdot p. \quad (387)$$

Conceive the number of sides by continual duplication to become indefinitely great. Then P has for limit $\odot ABE$, and $\text{rect. } r \cdot p$ has for limit $\text{rect. } R \cdot C$, C being the limit of p and R that of r ;

$$\therefore \odot ABE \approx \frac{1}{2} \text{ rect. } R \cdot C, \quad \text{Q.E.D.} \quad (236)$$

(they being the limits of variables always equal.)

SCHOLIUM. This theorem may be stated under the form:
The area of a circle is measured by one half the product of its radius and perimeter.

398. COR. 1. *The area of a circle is equal to π times the square of its radius.*

For denoting the area of the circle by S ,

since $S = \frac{1}{2} R \cdot C$, (397), and $C = 2\pi \cdot R$ (396),

$$S = \frac{1}{2} R \times 2\pi \cdot R = \pi R^2.$$

399. COR. 2. *The areas of circles are to each other as the squares of their radii.*

For $S = \pi \cdot R^2$, and $S' = \pi \cdot R'^2$; (398)

$$\therefore S : S' = \pi R^2 : \pi R'^2 = R^2 : R'^2.$$

400. DEFINITION. A *segment* of a circle is the figure bounded by an arc and its chord.

401. DEFINITION. A *sector* of a circle is the figure bounded by two radii and their intercepted arc.

402. DEFINITION. *Similar segments and sectors* in different circles are such as have arcs measuring equal angles at the center.

403. COR. 3. *The area of a sector is measured by one half the product of its radius and arc.*

For the sector is to the circle as the arc of the sector is to the circumference.

404. COR. 4. *Similar sectors are as the squares of their radii.*

For they are like parts of their respective circles.

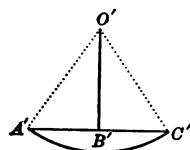
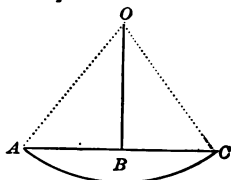
EXERCISE 575. What figure is both a segment and a sector?

576. The area of one circle is twice that of another. What is the ratio of their radii?

577. The radii of two similar segments is as 3 to 5. What is the ratio of their areas?

PROPOSITION XI. THEOREM.

405. *Similar segments are to each other as the squares of their radii.*



Given: $OA = R$, $O'A' = R'$, radii of similar segments B , B' ;

To Prove: $B : B' = R^2 : R'^2$.

Since $\angle O = \angle O'$, (Hyp.)

sector $OABC$ is similar to sector $O'A'B'C'$. (402)

Since $OA = OC$, $O'A' = O'C'$, and $\angle O = \angle O'$,

$\triangle OAC$ is similar to $\triangle O'A'C'$. (290)

Since sect. $OABC$: sect. $O'A'B'C' = R^2 : R'^2$, (404)

and $\triangle OAC : \triangle O'A'C' = \overline{OA}^2 : \overline{O'A'}^2 = R^2 : R'^2$, (342)

sect. $OABC$: $\triangle OAC =$ sect. $O'A'B'C'$: $\triangle O'A'C'$; (232''')

$\therefore OABC - OAC : OAC = O'A'B'C' - O'A'C' : O'A'C'$; (247)

\therefore seg. B : seg. $B' = \triangle OAC : \triangle O'A'C' = R^2 : R'^2$. Q.E.D. (244)

EXERCISE 578. In the diagram for Prop. XI., what must be the ratio of OA to $O'A'$ if segment B is $\frac{1}{2}$ of segment B' ?

579. In the same diagram, if the segment is $\frac{2}{3}$ of its circle, how many degrees are there in arc AC ?

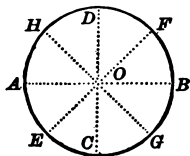
580. In the same diagram, if A were joined with the mid point D of arc AC , and $\angle CAD$ were found to be 38° , how many degrees in $\angle AOC$?

581. In the same diagram, if a line AF were drawn perpendicular to OA , and $\angle CAF$ were found to be 41° , how many degrees in arc AC ? How many in the angle formed by lines drawn from A and C to any point in arc AC ?

DIVISION OF CIRCUMFERENCE.

PROPOSITION XII. THEOREM.

406. *A circumference can be divided into 2, 4, 8, 16, ... equal arcs.*



Given: A circumference $ACBD$;

To Prove: $ACBD$ can be divided into 2, 4, 8, 16, ... equal arcs.

Find the center of $ACBD$. (173)

1°. Through O draw a diameter AB .

AB bisects the circumf. into the equal arcs ACB , ADB . (169)

2°. Through O draw a diam. $CD \perp$ to AB . (95)

Since the angles at O are equal, being rt. \angle s, (Const.)

$\text{arc } AC = \text{arc } CB = \text{arc } BD = \text{arc } DA = \frac{1}{4} ACBD$. (257)

3°. Bisect each of the \angle s at O by diameters EF , GH . (81)

The arcs AE , EC , CG , etc., each $= \frac{1}{2}$ a quad. $= \frac{1}{8} ACBD$. (Const.)

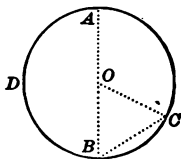
By successive bisections we can obtain, in the same way, the 16th, 32d, ... 2^n th part of $ACBD$, and hence can construct regular polygons of 4, 8, 16, ... 2^n sides. Q.E.D.

EXERCISE 582. In the diagram for Prop. XII., if A and E be joined with any point P in arc AE , how many degrees in the angle thus formed?

583. How many degrees in the angle formed by joining A and G with any point in arc ADB , and what is its ratio to the angle found in the preceding exercise?

PROPOSITION XIII. THEOREM.

407. *A circumference can be divided into 3, 6, 12, ... equal arcs.*



Given: A circumference $ACBD$;

To Prove: $ACBD$ can be divided into 3, 6, 12, ... equal arcs.

Find the center O (173), and draw any diameter AOB .

1°. From B draw a chord $BC = BO$ (178), and join CO .

Since $CO = BO = BC$, (Const.)

$$\angle B = \angle BOC. \quad (68)$$

But $\angle B$ is inscribed, and $\angle BOC$ is at the center;

$$\therefore \text{arc } AC = 2 \text{ arc } BC; \quad (268)$$

i.e., $\text{arc } AC = \frac{2}{3} \text{ arc } BCA$, a semicircumf.;

i.e., $\text{arc } AC = \frac{1}{3} ACBD$, the circumf.

$$\begin{aligned} 2^\circ. \quad & \text{Since arc } BC = \frac{1}{2} \text{ arc } AC, \\ & \text{arc } BC = \frac{1}{6} ACBD. \end{aligned} \quad (1^\circ)$$

3°. By bisecting arc BC again, we obtain $\frac{1}{12} ACBD$, thence

$$\frac{1}{24} ACBD, \text{ and so on to any arc denoted by } \frac{1}{3 \times 2^n} ACBD.$$

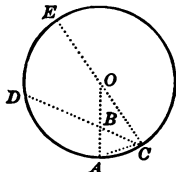
Q.E.D.

408. COR. *The side of a regular inscribed hexagon is equal to the radius of the circle.*

For BC , which subtends $\frac{1}{6} ACBD$, is equal to BO (Const.). We are thus enabled to construct any regular polygon of 3×2^n sides.

PROPOSITION XIV. THEOREM.

409. *A circumference can be divided into 5, 10, 20, ... equal arcs.*



Given: A circumference $ACED$;

To Prove: $ACED$ can be divided into 5, 10, 20, ... equal arcs.

Find the center O (173), and draw any radius OA .

1°. Divide OA in B , so that $OA : OB = OB : BA$. (308)

From A draw a chord $AC = OB$ (178), and draw CBD , to meet the circumference in D . Draw the diameter COE .

Since $OA : OB = OB : BA$,
and $AC = OB$, and $OC = OA$, } (Const.)
we have (1) $OA : AC = AC : BA$,
and (2) $OC : AC = OB : BA$.

From (1), since $\angle A$ is com. to both \triangle ,

$\triangle BAC$ is similar to $\triangle CAO$; (290)

$\therefore \angle ACB$ or $\angle ACD = \angle AOC$. (289)

From (2), since the sides of $\triangle CAO$ are as the segments of the base OA , $\angle ACB = \angle OCB$; (279)

\therefore arc $ED =$ arc AD . (257)

But arc $AD = 2$ arc AC , (268)

(since $\angle ACD$ is inscribed and $\angle AOC$ is at the center;)

\therefore arc $AD +$ arc $DE +$ arc $AC = 5$ arc AC ;

\therefore arc $AD = \frac{2}{3}$ arc $CDE = \frac{1}{3}$ arc $ACED$.

2°. Since arc $AC = \frac{1}{3}$ arc AD , (1°)

arc $AC = \frac{1}{10}$ arc $ACED$.

3°. By bisecting AC we obtain $\frac{1}{20} ACED$, and so on. Thus we can divide the circumference $ACED$ into 5, 10, 20, ... 5×2^n equal arcs. Q.E.D.

410. COR. *By taking the difference of two arcs respectively equal to $\frac{1}{6}$ and $\frac{1}{10}$ of the circumference, since $\frac{1}{6} - \frac{1}{10} = \frac{1}{15}$, we can find an arc that is $\frac{1}{15}$ of the circumference; and thence, by repeated bisections, arcs that are $\frac{1}{30}, \frac{1}{60}, \dots$ of the circumference.*

Till the beginning of this century, the division of the circumference by means of the straight line and circle could be effected only in the cases covered by the preceding propositions; that is, the circumference could be thus divided into $1 \times 2^n, 3 \times 2^n, 5 \times 2^n, 15 \times 2^n$, equal arcs. The celebrated geometer Gauss proved, however, that the circumference can be divided into any number of equal arcs expressed by the formula $2^n + 1$ when this is a prime number. Thus, as we have seen, we can effect the division into $2^0 + 1 = 2, 2^1 + 1 = 3, 2^2 + 1 = 5$, equal arcs, while $2^3 + 1 = 9$, not being a prime number, does not come under the rule. The division into $2^4 + 1 = 17$ equal arcs, not to speak of greater numbers, involves a process too intricate to come within the scope of an elementary work.

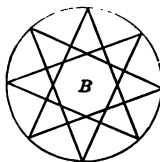
411. STELLATE POLYGONS. If a circumference be divided into n equal parts, and, beginning at one of the points of division, we go round the circumference drawing chords subtending these arcs, m and m , m being prime to n and $< \frac{1}{2}n$, we obtain a polygon of starlike form called a *stellate polygon*.

. We say m is to be $< \frac{1}{2}n$, so as to confine our discussion to the minor of the arcs subtended by each chord, and consisting respectively of m and $n - m$ of the equal parts. If m were a factor of n , say $am = n$, then we should return to the starting point after placing a chords, forming a regular polygon of a sides. But, if m is prime to n , then as mn is the least number that divided by m gives n as quotient, we shall go round the circumference m times before returning to the starting point, and in doing so shall place n chords.

There is, accordingly, no stellate polygon of 3, 4, or 6 vertices, since each of these numbers has no prime to it less than its half.

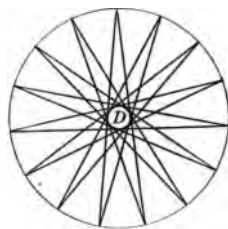
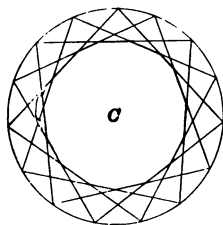
Since 2 is the only number prime to 5 and less than its half, there is but one stellate pentagon; as *A*.

Since 3 is the only number prime to 8 and less than its half, there is but one stellate octagon; as *B*.



For the like reason there is but one stellate decagon, and one stellate dodecagon.

But since there are three numbers, 2, 4, and 7, that are prime to 15 and less than its half, there are three stellate pentadecagons, of which two, *C* and *D*, are given.

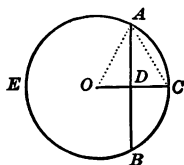


In like manner, as there are seven numbers, 8, 7, 6, 5, 4, 3, 2, that are prime to 17 and less than its half, there are seven stellate polygons having 17 vertices.

Among the many methods that have been devised for obtaining an approximate value of the important constant denoted by π , that based upon the following proposition is perhaps the simplest, enabling us to find, in succession, the value of a side of a polygon of $2n$, $4n$, ... sides.

PROPOSITION XV. THEOREM.

412. *The radius and the value of the chord of an arc being given, we can find the value of the chord of half that arc.*



Given : In circle $ACBE$, a radius OC perpendicular at D to AB , the chord of arc ACB ;

To Prove : The value of the chord of one half arc ACB can be found.

Since OC is \perp to AB at D , (Hyp.)

$$AD = \frac{1}{2} AB, \text{ and arc } AC = \frac{1}{2} \text{ arc } ACB. \quad (172)$$

Join OA, AC . Then since $AC < a$ quadrant,*

$$\angle AOC < a \text{ rt. } \angle ;$$

$$\therefore \overline{AC}^2 \approx \overline{OA}^2 + \overline{OC}^2 - 2 OC \cdot OD ; \quad (350)$$

$$\text{i.e., } \overline{AC}^2 \approx 2 R^2 - 2 R \cdot OD.$$

Since $\angle ADO$ is a rt. \angle ,

(Hyp.)

$$\overline{OD}^2 \approx \overline{OA}^2 - \overline{AD}^2 \approx R^2 - \frac{1}{4} \overline{AB}^2 ; \quad (347)$$

$$\therefore OD = \sqrt{R^2 - \frac{1}{4} AB^2} = \frac{1}{2} \sqrt{4 R^2 - AB^2},$$

(OD, R, AB , here denoting numerical measures;)

$$\therefore \overline{AC}^2 = 2 R^2 - 2 R \cdot \frac{1}{2} \sqrt{4 R^2 - AB^2} ;$$

$$\therefore AC = \sqrt{2 R^2 - R \sqrt{4 R^2 - AB^2}}.$$

If, as is usually done, we take $R = 1$, then

$$AC = \sqrt{2 - \sqrt{4 - AB^2}}.$$

* At most $AC = \frac{1}{4}$ of the circumference, since $ACB =$, at most, $\frac{1}{2}$ of the circumference.

PROPOSITION XVI. PROBLEM.

413. *To find an approximate value of π , the ratio of circumference to diameter.*

$$\text{FORMULA: } s_{2n} = \sqrt{2 - \sqrt{4 - s_n^2}}$$

where s_n denotes the value of a side of a regular inscribed polygon of n sides, R being taken = 1. If we take $n = 6$, then $s_n = 1$ (408).

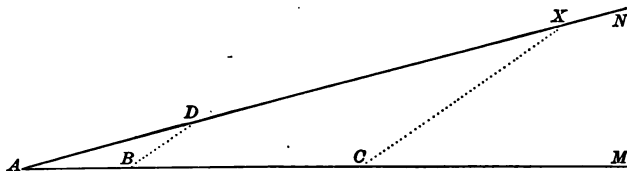
COMPUTATION.	VAL. OF SIDE.	VALUE OF PERIM. = $n.s.$
$s_{12} = \sqrt{2 - \sqrt{4 - 1}}$	= .51763809.	6.21165708.
$s_{24} = \sqrt{2 - \sqrt{4 - (.51763809)^2}}$	= .26105238.	6.26525722.
$s_{48} = \sqrt{2 - \sqrt{4 - (.26105238)^2}}$	= .13080626.	6.27870041.
$s_{96} = \sqrt{2 - \sqrt{4 - (.13080626)^2}}$	= .06543817.	6.28206396.
$s_{192} = \sqrt{2 - \sqrt{4 - (.06543817)^2}}$	= .03272346.	6.28290510.
$s_{384} = \sqrt{2 - \sqrt{4 - (.03272346)^2}}$	= .01636228.	6.28311544.
$s_{768} = \sqrt{2 - \sqrt{4 - (.01636228)^2}}$	= .00818126.	6.28316954.

Taking this last value of the perimeter as an approximation towards the value of the circumference whose radius is 1, since $\pi = \frac{C}{2R}$ (396), we obtain $\pi = \frac{1}{2}(6.28317) = 3.14159$ nearly, a result correct to the last decimal.

By the useless expenditure of much time and labor, the value of π has been calculated as far as 700 places of decimals. The first fifteen, more than sufficient for any useful purpose, are $\pi = 3.141592653589793$. Ten decimals are sufficient to give the circumference of the earth to the fraction of an inch, and thirty decimals would give the circumference of the whole visible universe to a quantity imperceptible with the most powerful microscope.

PROPOSITION XVII. PROBLEM.

414. *The diameter being given, to find a line approximately equal to the circumference.*



Given : D , the diameter of a circle ;

Required : To find a straight line approximately equal to the circumference of that circle.

Draw indefinite lines AM , AN , making any angle.

Upon AM lay off $AB=113$, and $AC=355$, units of length,* and upon AN lay off $AD=D$. Join BD and draw $CX \parallel$ to BD .

$$\text{Since } AB : AC = AD : AX = 113 : 355, \quad (274)$$

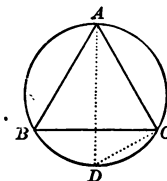
$$AX = \frac{355}{113} AD = D \times 3.141592 \dots \quad \text{Q.E.F.}$$

The line thus obtained is greater than the circumference by about one four millionth of the diameter, as the student may easily prove for himself.

415. In the inscribed equilateral triangle ABC , draw $AD \perp$ to BC , and join CD . Then AD bisects arc BDC (172), and $DC = R$, the radius (408) ;

$$\therefore \overline{AC}^2 = \overline{AD}^2 - \overline{DC}^2 = 4R^2 - R^2 = 3R^2 ;$$

$$\therefore AC = R\sqrt{3}.$$

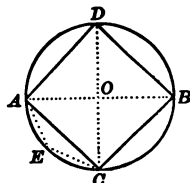


* Lengths respectively equal to 1.13 and 3.55 units can be taken from a diagonal scale ; or see Exercise 836.

416. In the inscribed square $ADBC$, joining AB , CD ,

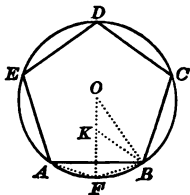
$$\overline{AC}^2 = \overline{AO}^2 + \overline{OC}^2 = 2R^2 \quad (347);$$

$$\therefore AC = R\sqrt{2}.$$



417. In the inscribed pentagon, the relation of the side to the radius is most readily obtained from the expression for the side of the inscribed decagon (see 420). From this we obtain

$$AB = \frac{1}{2} R \sqrt{10 - 2\sqrt{5}}.$$



418. In the inscribed hexagon, the side $DC = R$ (408), a result that can be obtained also by means of the formula referred to in the next article.

419. In the inscribed octagon (see diagram for Art. 416) the value of a side is most readily found by means of the formula obtained in Prop. XV.; that is

$$AC = \sqrt{2R^2 - R\sqrt{4R^2 - AB^2}}.$$

In this, putting AE for AC , and $R\sqrt{2}$, the value of a side of the inscribed square, for AB , we obtain

$$AE = \sqrt{2R^2 - R\sqrt{4R^2 - 2R^2}};$$

$$\text{i.e., } AE = \sqrt{2R^2 - R^2\sqrt{2}} = R\sqrt{2 - \sqrt{2}}.$$

420. In the inscribed decagon $AF \dots E$ (see diagram for Art. 417),

$$\text{since } R : BF = BF : R - BF, \quad (409)$$

$$\overline{BF}^2 = R^2 - R \cdot BF; \quad (322)$$

$$\therefore BF = \frac{1}{2} R(\sqrt{5} - 1), \text{ solving as a quadratic.}$$

In the formula $AC = \sqrt{2R^2 - R\sqrt{4R^2 - AB^2}}$, putting $\frac{1}{2}R(\sqrt{5} - 1)$, the value of a side of the inscribed decagon, for AC , and squaring, we obtain

$$\frac{1}{4}R^2(\sqrt{5} - 1)^2 = 2R^2 - R\sqrt{4R^2 - AB^2};$$

$$\text{whence } 2\sqrt{4R^2 - AB^2} = R(\sqrt{5} + 1);$$

$$\text{whence } AB = \frac{1}{2}R(10 - 2\sqrt{5}).$$

EXERCISES.

QUESTIONS.

584. Into how many equal parts, up to 15, are we able to divide a circumference by means of the straight line and the circle?

585. What is the ratio of the radius of the circle circumscribed about an equilateral triangle to that of the inscribed circle?

586. By which of the constructions of this book can a right angle be divided into five equal parts?

587. How many different stellate polygons can be formed if we have a circumference divided into 7 equal parts? 9 equal parts? 11 equal parts? 13 equal parts?

588. If two nonadjacent sides of a regular pentagon be produced to meet, what angle will they contain?

589. What is the value of the interior angles of a regular octagon? Of a regular decagon? Of a regular pentadecagon? What is the limit towards which each of the interior angles of a regular polygon tends as the number of sides increases indefinitely? What is the limit for each exterior angle?

590. What is the circumference and area of a circle whose diameter is 10 inches, supposing $\pi = 3.1416$?

591. What is the circumference and area of a circle whose radius is 2 ft. 6 in.?

592. What is the circumference of a circle whose area is 100 sq. in.?

593. What is the radius of a circle whose area is 6 sq. ft. ?
594. What is the radius of a circle whose circumference is 12 in. ?
595. What is the radius of a circle whose area is n times that of a circle with radius R ?
596. What is the area of the ring between two concentric circles whose radii are 5 ft. and 7 ft. respectively ?
597. Within a circle whose radius is R , a circle is drawn so as to cover $\frac{1}{2}$ of the surface of the first circle. What is the radius of the second circle ?
598. The radii of two similar segments are as 4 to 7 ; if the first segment contains 25 sq. in., what does the other contain ?
599. In a white circle of 3 in. radius, an inscribed square is painted black. How much white surface will remain ?
600. In a white square whose side is 4 in., an inscribed circle is painted red. How much white surface will remain ?
601. If the radius of a circle is 10 in., what is the side of the inscribed equilateral triangle ?
602. If the side of an inscribed equilateral triangle is 10 in., what is the radius of the circle ?
603. If the radius of a circle is 8 in., what is the side of a regular inscribed pentagon ?
604. If the side of a regular inscribed pentagon is 9 in., what is the radius of the circle ?
605. If the radius of a circle is 6 in., what is the side of a regular inscribed octagon ?
606. What must be the radius of a circle so that a side of a regular inscribed octagon shall be 10 in. ?
607. If the radius of a circle is 10 in., what is the side of a regular inscribed decagon ?
608. What must be the radius of a circle so that a side of a regular inscribed decagon shall be 3 in. ?

THEOREMS.

609. An angle of a regular polygon of n sides is to an angle of a regular polygon of $n + 2$ sides, as $n^2 - 4$ is to n^2 .
610. If the bisectors of all the angles of a polygon meet in a point, a circle can be inscribed in that polygon.

611. An inscribed equilateral triangle is equivalent to half the regular hexagon of the same radius.

612. The altitude of an equilateral triangle is equal to the side of an equilateral triangle inscribed in a circle whose diameter is the base of the first.

613. The altitude of an equilateral triangle is to the radius of the circumscribing circle as 3 is to 2.

614. The area of the regular hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

615. The square of a side of an inscribed equilateral triangle is equivalent to three times the square of a regular hexagon inscribed in the same circle.

616. If the arcs subtended by two sides of an equilateral triangle be bisected, the chord joining those points will be trisected by those sides.

617. The diagonals drawn from a vertex of a regular pentagon to the opposite vertices, trisect that angle.

618. The diagonals drawn from a vertex of any regular polygon of n sides to the opposite vertices, divide the angle into $n - 2$ equal parts.

619. The diagonals joining alternate vertices of a regular pentagon form by their intercepts a regular pentagon.

620. If the alternate sides of a regular pentagon be produced to meet, the points of meeting will be the vertices of another regular pentagon.

621. The intersecting diagonals of a regular pentagon divide each other in extreme and mean ratio.

622. In a regular pentagon $ABCDE$, diagonals AC , BE , are drawn, intersecting in F ; show that FD is a parallelogram.

623. A ribbon is folded into a flat knot of five edges; show that these edges form a regular pentagon.

624. If P , H , and D denote respectively a side of a regular inscribed pentagon, hexagon, and decagon; then $P^2 \approx H^2 + D^2$.

625. If from any point within a regular polygon of n sides, perpendiculars be drawn to the sides, the sum of these perpendiculars will be equal to n times the apothem.

626. The radius of an inscribed regular polygon is a mean proportional between its apothem and the radius of the similar circumscribed polygon.

627. The area of a circular ring, *i.e.*, the space between two concentric circumferences, is equal to that of a circle having for diameter a chord of the outer circle tangent to the inner circle.

628. If, on the hypotenuse and the arms of a right triangle as diameters, semicircles be described, the curvilinear figures bounded by the greater and the two lesser semicircumferences will be equivalent to the triangle.

PROBLEMS.

Inscribe in a given circle :

629. An equilateral triangle.

631. A square.

630. A regular pentagon.

632. A regular octagon.

Circumscribe about a given circle :

633. An equilateral triangle.

635. A regular pentagon.

634. A regular hexagon.

636. A regular decagon.

637. Describe a square about a given rectangle.

638. Inscribe an equilateral triangle in a given square, so as to have a vertex of the triangle at a vertex of the square.

639. Construct an equilateral triangle that shall be double the area of a given equilateral triangle.

640. Construct a square that shall be $\frac{3}{4}$ of a given square.

641. Construct a regular pentagon that shall be $\frac{3}{4}$ of a given regular pentagon.

642. Construct a regular hexagon that shall be $\frac{4}{5}$ of a given regular hexagon.

643. Describe a circle equivalent to $\frac{5}{8}$ of a given circle.

PART II. SOLID GEOMETRY.



BOOK VII.

PLANES AND POLYHEDRAL ANGLES.

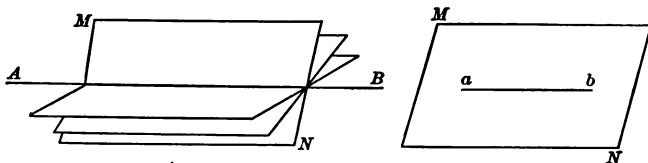


PLANES AND PERPENDICULARS.

Thus far have been investigated the properties of figures confined to one plane. We are now prepared to enter upon the properties of figures in space; hence the name, *Geometry of Space*, often applied to this department of the subject, also called *Solid Geometry*, and *Geometry of Three Dimensions*. It is to be remembered that, although in our diagrams we can represent only a limited portion of a plane, the plane thus represented is to be regarded as having indefinite extension.

PROPOSITION I. THEOREM.

421. *Through any given straight line an infinite number of planes can be passed.*



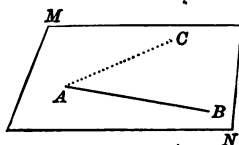
For if we take any two points a, b , in a plane MN , the

straight line drawn joining those points will be wholly in that plane (9). By making the line ab , thus drawn, to coincide with any given line AB , we have one plane MN passing through AB . By revolving MN round AB as an axis, MN can be made to occupy any number of positions, each of which is the position of a plane passing through AB .

SCHOLIUM. Hence a plane is not determined—that is, marked off from other planes—by the single condition that it passes through a given straight line. To determine a plane, some other condition must be given.

PROPOSITION II. THEOREM.

422. *A plane is determined by a straight line and a point without that line.*



Given: A straight line AB , and a point C not in AB ;

To Prove: Only one plane can pass through AB and also through C .

A plane MN being passed through, and revolved about, AB (421) will have a determined position when it comes to contain the point C . For if it be then turned in either direction about AB , it will cease to contain the point C . Q.E.D.

423. COR. 1. *A plane is determined by two straight lines, intersecting or parallel.*

For it will be determined by either of those lines and any point without it in the other line (422).

424. COR. 2. *A plane is determined by any three points not in the same straight line.*

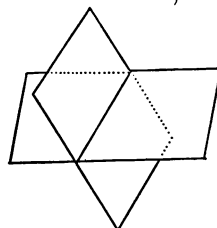
For any two of the points may be joined by a straight line, which, with the third point, determines the plane (422).

425. COR. 3. *Through a given point in space can pass only one parallel to a given straight line.*

For in the one plane determined by the line and the point, there can pass through that point only one parallel to the given line (105).

426. COR. 4. *The intersection of two planes is a straight line.*

For their intersection cannot contain three points that are not in the same straight line, seeing that one plane and only one can pass through three such points (424).



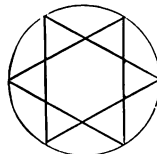
427. DEFINITIONS. A straight line is *perpendicular to a plane*, if it is perpendicular to every straight line drawn in the plane through its *foot*, the point in which it meets the plane. The plane, in this case, is also said to be *perpendicular to the line*.

EXERCISE 644. In the diagram for Prop. II., if a plane were passed through AC , and turned about that line as an axis, when would the revolving plane coincide with plane MN ? If the plane continued to revolve, when would it again coincide with MN ?

645. The intersection of a plane with a surface on which no straight line can be drawn, *e.g.* the surface of an eggshell or of an apple, is what sort of a line?

646. Enunciate the converse of Cor. 4 of Prop. II.; is that converse generally true?

647. If a circumference be divided into six equal parts, and chords be drawn subtending these parts, two and two, as in the accompanying diagram, is the six-pointed figure thus formed a stellate polygon according to the definition given in Art. 411? If not, why not?

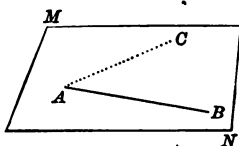


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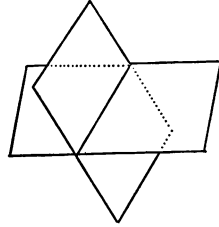
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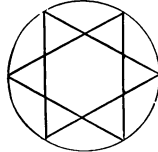
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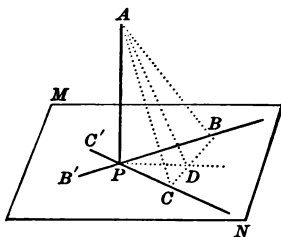
646. Enunciate the converse of Cor. 4 of Prop. II.; is that converse generally true?

647. If a circumference be divided into six equal parts, and chords be drawn subtending these parts, two and two, as in the accompanying diagram, is the six-pointed figure thus formed a stellate polygon according to the definition given in Art. 411? If not, why not?



PROPOSITION III. THEOREM.

428. *A straight line perpendicular to each of two straight lines at their intersection, is perpendicular to the plane of those lines.*



Given: AP perpendicular to both BB' and CC' in plane MN , at P ;

To Prove: AP is perpendicular to MN at P .

Through P draw any straight line PD in MN ; and through D draw BDC so that $DB = DC$ (219). Join A with B , D , and C .

Since AD , PD , are medians of $\triangle ABC$, PBC , resp., (Const.)

$$\left. \begin{aligned} \overline{AB}^2 + \overline{AC}^2 &\approx 2 \overline{AD}^2 + 2 \overline{BD}^2, \\ \text{and } \overline{PB}^2 + \overline{PC}^2 &\approx 2 \overline{PD}^2 + 2 \overline{BD}^2; \end{aligned} \right\} \quad (353)$$

$$\therefore \overline{AB}^2 - \overline{PB}^2 + \overline{AC}^2 - \overline{PC}^2 \approx 2 \overline{AD}^2 - 2 \overline{PD}^2. \quad (\text{Ax. 3})$$

But $\triangle APB$ and $\triangle APC$ are rt. \triangle ; (Hyp.)

$$\therefore \overline{AB}^2 - \overline{PB}^2 \approx \overline{AP}^2, \text{ and } \overline{AC}^2 - \overline{PC}^2 \approx \overline{AP}^2; \quad (347)$$

$$\therefore 2 \overline{AP}^2 \approx 2 \overline{AD}^2 - 2 \overline{PD}^2;$$

$$\therefore \overline{AP}^2 + \overline{PD}^2 \approx \overline{AD}^2; \quad (\text{Ax. 2, Ax. 7})$$

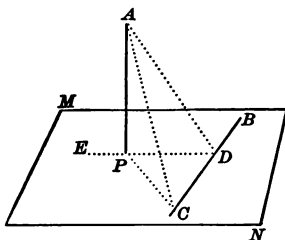
$$\therefore AP \text{ is } \perp \text{ to } PD; \quad (352)$$

$$\therefore AP \text{ is } \perp \text{ to } MN, \quad \text{Q.E.D.} \quad (427)$$

(being \perp to any line in MN through P .)

PROPOSITION IV. THEOREM.

429. *A perpendicular to a plane can be drawn from any given point without or in that plane.*



1°. *Given:* A point A without a plane MN ;

To Prove: A perpendicular to MN can be drawn from A .

In the plane passed through A and any line BC in MN , draw $AD \perp$ to BC . From D draw, in the plane MN , $DE \perp$ to BC ; and, in the plane of AD , DE , draw $AP \perp$ to DE . Then AP is \perp to MN .

For since ADC , PDC , and APD are rt. Δ s, (Const.)

$$\left. \begin{aligned} \overline{AC}^2 &\approx \overline{AD}^2 + \overline{DC}^2, \\ \text{and } \overline{PC}^2 &\approx \overline{PD}^2 + \overline{DC}^2; \end{aligned} \right\} \quad (347)$$

$$\therefore \overline{AC}^2 - \overline{PC}^2 \approx \overline{AD}^2 - \overline{PD}^2 \approx \overline{AP}^2; \quad (\text{Ax. 3, 347})$$

$$\therefore \overline{AC}^2 \approx \overline{AP}^2 + \overline{PC}^2; \quad (\text{Ax. 2})$$

$$\therefore AP \text{ is } \perp \text{ to } PC; \quad (352)$$

$$\therefore AP \text{ is } \perp \text{ to } MN, \quad \text{Q.E.D.} \quad (428)$$

(being \perp to both PD and PC .)

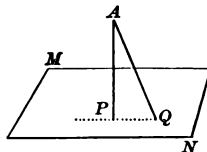
2°. *Given:* A point P in plane MN ;

To Prove: A perpendicular to MN can be drawn at P .

From P draw $PD \perp$ to BC , any line in MN ; and in any plane intersecting MN in BC , draw $DA \perp$ to BC . Then in

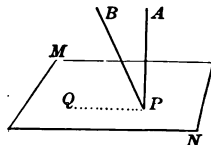
the plane of PD , DA , draw $PA \perp$ to PD . PA is \perp to MN , as may be proved in the same way as in 1°.

430. COR. 1. *There can be drawn but one perpendicular to a given plane from a point without the plane.*



For if there could be two perpendiculars, AP , AQ , from A to MN , draw PQ . Then, in the plane APQ , there would be two perpendiculars from A to the same line PQ , which is impossible (51).

431. COR. 2. *At a given point in a plane there can be drawn but one perpendicular to that plane.*



For if there could be two perpendiculars, PA , PB , from P in MN , suppose a plane to pass through PA , PB , intersecting MN in PQ . Then from the same point in PQ , and in the same plane, there would be two perpendiculars to PQ , which is impossible (41).

432. COR. 3. *The perpendicular is the shortest line that can be drawn to a plane from a given point without the plane.*

For AP is shorter than any oblique line, AQ , drawn from A to PQ , in the same plane APQ (93).

433. DEFINITION. *The distance of a point from a plane is the length of the perpendicular from the point to the plane.*

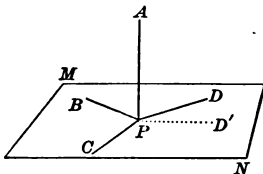
EXERCISE 648. In the diagram for Prop. III., the planes determined by AB and P , AC and P , AD and P , intersect in what line?

649. In those planes, if circumferences be described on AB , AC , and AD , as diameters, they will pass through what point and have what common chord?

650. In the diagram for Prop. V., if AB and AD be joined, and $PD = AB$, show that $(AD + AP)(AD - AP) = AB^2$.

PROPOSITION V. THEOREM.

434. *All perpendiculars drawn to a given straight line at a given point, lie in a plane perpendicular to the line.*



Given: PB, PC, PD , any perpendiculars to AP at P ;

To Prove: PB, PC, PD , all lie in a plane that is \perp to AP at P .

Through PB, PC , pass the plane MN .

Since AP is \perp to PB and PC , (Hyp.)

AP is \perp to MN , and MN is \perp to AP . (428, 427)

Through AP, PD , suppose a plane passed cutting MN in PD' .

Since AP is \perp to MN ,

AP is \perp to PD' , a line in MN . (427)

But AP is \perp to PD ; (Hyp.)

$\therefore PD$ must coincide with PD' and lie in MN , Q.E.D.

(since, in plane AP, PD , there can be but one \perp to AP at P .)

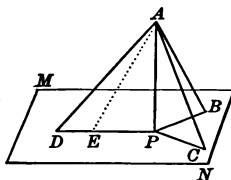
(41)

435. COR. 1. *If an indefinite line PB be made to revolve so as to remain always perpendicular to AP at P , it will generate the plane MN at right angles to AP .*

436. COR. 2. *Through a given point in or without a straight line can be passed but one plane perpendicular to the line.*

PROPOSITION VI. THEOREM.

437. *Obliques drawn from a point so as to meet a plane at equal distances from the perpendicular, are equal; of obliques meeting the plane at unequal distances from the perpendicular, the more remote is the greater.*



- *Given:* AB, AC, AD , obliques drawn from A to MN , so that B and C are equally distant from the perpendicular AP , but PD is greater than PB ;

To Prove: AB is equal to AC , but AB is less than AD .

On PD lay off $PE = PB$, and join AE .

Since $PB = PE$, AP is common, } (Hyp. and Const.)
and the \angle at P are rt. \angle s,

$$AB = AC = AE. \quad (66, 70)$$

$$\text{But } PE < PD; \quad (\text{Const.})$$

$$\therefore AE < AD; \quad (99)$$

$$\therefore AB < AD. \quad \text{Q.E.D.}$$

438. COR. 1. *Conversely, equal obliques drawn from a point to a plane meet it at equal distances from the perpendicular; and, of unequal obliques from the same point, the greater meets the plane at a greater distance from the perpendicular than the less.*

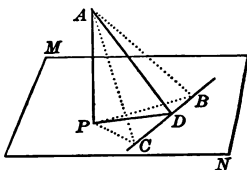
439. COR. 2. *The locus of all the points in a plane that are equally distant from a given point without the plane, is a circumference whose center is the foot of the perpendicular from the given point.*

440. COR. 3. *The locus of all the points in space that are equally distant from two given points, is the plane that is perpendicular at its mid point to the straight line joining the given points.*



PROPOSITION VII. THEOREM.

441. *If from the foot of a perpendicular to a plane a line be drawn at right angles to any line in the plane, and the foot of this be joined with any point in the first perpendicular, this line will be perpendicular to the line of the plane.*



Given: AP perpendicular to plane MN ; PD perpendicular to BC , any line in MN , at D ; and AD joining any point in AP with D ;

To Prove: AD is perpendicular to BC .

Lay off $DB = DC$, and join A and P each with B and C .

Since $DB = DC$ (Const.), PD is common,

and $\text{rt. } \angle PDB = \text{rt. } \angle PDC$, (Hyp.)

$PB = PC$; (66, 70)

$\therefore AB = AC$; (437)

$\therefore AD$ is \perp to BC , Q.E.D. (75)

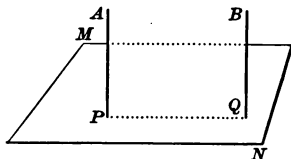
(A and D being points equally distant from B and C .)

442. COR. *If from a point without a plane two perpendiculars be drawn, one to the plane and the other to any line in the plane, the line joining the feet of these perpendiculars will be perpendicular to the line in the plane; i.e., if AP is \perp to MN and AD is \perp to BC , then PD is \perp to BC .*

PLANES AND PARALLELS.

PROPOSITION VIII. THEOREM.

443. *Two perpendiculars to the same plane are parallel.*



Given: AP and BQ , each perpendicular to plane MN ;

To Prove: AP is parallel to BQ .

Through AP , BQ , pass a plane APB meeting MN in PQ .

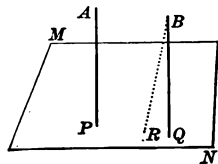
Since AP and BQ are each \perp to MN , (Hyp.)

AP and BQ are each \perp to PQ in plane APB ; (427)

$\therefore AP$ is \parallel to BQ . Q.E.D. (106)

444. COR. 1. *If AP , one of two parallels, is \perp to a plane MN , then BQ , the other, is also \perp to MN .*

For if we draw $BR \perp$ to MN (429), since BQ is \parallel to AP (Hyp.), and BR is \parallel to AP (443), BQ must coincide with BR and be \perp to MN , since otherwise there would be two \parallel s to AP through the same point B , which is impossible (425).



445. COR. 2. *If two parallels, A and B , are each parallel to a third line C , in another plane, they are parallel to each other.*

For if we pass a plane \perp to C , it will also be \perp to A and B (444); whence A and B are parallel (443).

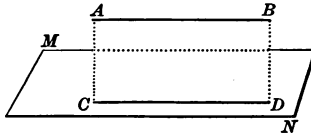
446. DEFINITION. A straight line is *parallel to a plane* if it cannot meet the plane though both be indefinitely produced.

447. DEFINITION. Two planes are *parallel* if they cannot meet though indefinitely produced.



PROPOSITION IX. THEOREM.

448. *Any straight line without a plane is parallel to the plane if parallel to any line in it.*



Given: AB parallel to CD , a line in plane MN ;

To Prove: AB is parallel to plane MN .

Pass a plane through AB and CD .

Since AB is in the plane AD , if it could meet the plane MN it would do so in the intersection of AD with MN , that is, in line CD .

But AB cannot meet CD , since they are parallel. (Hyp.)

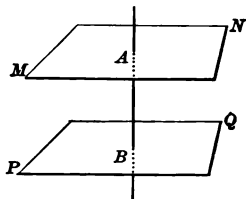
Hence AB cannot meet MN , i.e., AB is \parallel to MN . Q.E.D. (446)

449. COR. *If a line AB is parallel to a plane MN , and a plane be passed through AB so as to intersect MN in CD , then CD is parallel to AB .*

For AB , though in the same plane with CD , cannot meet it, seeing that CD lies in MN , which AB cannot meet.

PROPOSITION X. THEOREM.

450. *Planes that are perpendicular to the same straight line are parallel.*



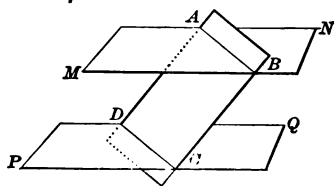
Given: Planes MN and PQ , each perpendicular to AB ;

To Prove: Plane MN is parallel to plane PQ .

For if MN could meet PQ , then through any point common to both would be passed two planes perpendicular to the same straight line AB . But this is impossible (436). Hence MN cannot meet PQ . Q.E.D.

PROPOSITION XI. THEOREM.

451. *The intersections of two parallel planes with a third plane are parallel.*



Given: Two parallel planes, MN , PQ , intersecting plane BD in AB , DC ;

To Prove: AB is parallel to DC .

Since plane MN cannot meet plane PQ , (Hyp.)
 AB cannot meet DC , though in the same plane BD ;
 $\therefore AB$ is \parallel to DC . Q.E.D. (102)

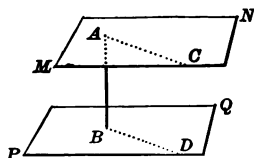
452. COR. *Parallel lines that are intercepted between parallel planes are equal.*

For since the plane of the parallels AD , BC , intersects the parallel planes MN , PQ , in parallel lines AB , DC , the figure AC is a parallelogram (131); whence $AD = BC$ (136).



PROPOSITION XII. THEOREM.

453. *A straight line perpendicular to one of two parallel planes is perpendicular to the other also.*



Given: Two parallel planes, MN , PQ , and a straight line AB perpendicular to MN ;

To Prove: AB is perpendicular to PQ .

Through AB pass any plane AD , intersecting MN in AC , and PQ in BD .

Since plane MN is \parallel to plane PQ , (Hyp.)

AC is \parallel to BD ; (451)

$\therefore AB$ is \perp to AC and BD ; (Hyp., 107)

$\therefore AB$ is \perp to plane PQ , Q.E.D. (427)

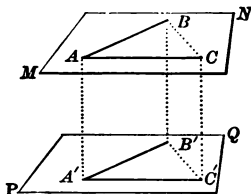
(being \perp to any line through B in PQ .)

454. COR. *Through a given point, A , one plane can be passed parallel to a given plane, PQ , and only one.*

For from A , a perpendicular AB can be drawn to PQ (429); and through A , a plane can be passed \perp to AB , and hence \parallel to plane PQ (450). Moreover, since from A but one perpendicular can be drawn to PQ (430), there can be but one plane passed through A \parallel to PQ .

PROPOSITION XIII. THEOREM.

455. *If two angles not in the same plane have their sides respectively parallel and drawn in the same direction, they are equal.*



Given: Two angles, BAC , $B'A'C'$, lying in planes MN and PQ , respectively, so that BA and $B'A'$, CA and $C'A'$, are respectively parallel and drawn in the same direction;

To Prove: Angle BAC is equal to angle $B'A'C'$.

In AB , AC , take any points B and C , and lay off $A'B' = AB$, $A'C' = AC$; join AA' , BB' , CC' .

Since AB , AC , are resp. \parallel and $=$ to $A'B'$, $A'C'$,
(Hyp. and Const.)

AB' and AC' are parallelograms; (142)

BB' , CC' , are each \parallel and $=$ to AA' ; (136)

$\therefore BB'$ is \parallel and $=$ to CC' ; (445, Ax. 1)

$\therefore BC$ is \parallel and $=$ to $B'C'$;

$\therefore \triangle BAC = \triangle B'A'C'$; (69)

$\therefore \angle A = \angle A'$. Q.E.D. (70)

456. COR. 1. *If two angles lying in different planes have their sides respectively parallel, their planes are parallel.*

For the intersecting lines that determine the one plane, being parallel to the intersecting lines that determine the other, the planes are parallel.

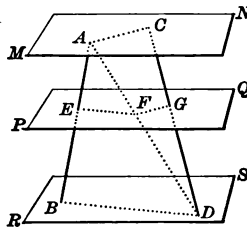
457. COR. 2. *If two parallel planes, MN and PQ , are intersected by two other planes, AB' , AC' , the angles A , A' , formed by their intersections, are equal.*

458. COR. 3. *If three lines, AA' , BB' , CC' , not in one plane, are equal and parallel, the triangles ABC , $A'B'C'$, formed by joining their extremities, are equal, and their planes are parallel.*



PROPOSITION XIV. THEOREM.

459. *If two straight lines are cut by three parallel planes, the intercepts are proportional.*



Given: A line AB meeting parallel planes MN , PQ , RS , in A , E , B , respectively; and a line CD meeting the same planes in C , F , D , respectively;

To Prove: $AE : EB = CF : FD$.

Draw AD , meeting PQ in F ; join AC , EF , FG , and BD .

Since planes PQ , RS , are \parallel , and plane ABD cuts them,

$$EF \text{ is } \parallel \text{ to } BD; \quad (451)$$

$$\therefore AE : EB = AF : FD. \quad (274)$$

Since planes PQ , MN , are \parallel , and plane DAC cuts them,

$$FG \text{ is } \parallel \text{ to } AC; \quad (451)$$

$$\therefore CG : GD = AF : FD; \quad (274)$$

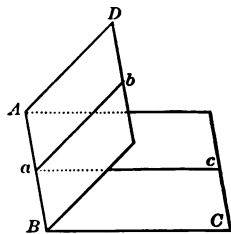
$$\therefore AE : EB = CG : GD. \quad (232''')$$

460. COR. *If n straight lines are cut by m parallel planes, the intercepts are proportional.*

DIHEDRAL ANGLES.

461. A *dihedral angle* is the opening between two planes that meet. The line in which the planes meet is called the *edge* of the angle, and the two planes are called its *faces*. Thus the faces AC , BD , meeting in the edge AB , contain the dihedral angle $DABC$.

To designate a dihedral angle, four letters are generally necessary, two at the edge and one on each face, the two at the edge being placed between the other two. If the edge belongs to only one angle, the letters at the edge will suffice to designate the angle. Thus the dihedral angle $DABC$ may be referred to as *dihedral angle* AB , or simply as *the dihedral* AB .



462. The *plane angle* of a dihedral angle is the angle contained by the two perpendiculars drawn, one in each face, to any point in the edge. Thus bac is the plane angle of the dihedral $DABC$. It is evident that the plane angle is the same at whatever point of the edge it is constructed (455).

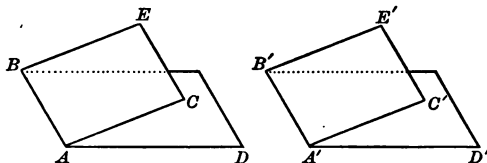
A dihedral angle may be conceived as *generated* by a plane BD turning from coincidence with plane AC about the edge AB as axis, till it reaches the position where its plane angle is $\angle bac$; which, again, may be conceived as generated by the revolution of the line ab from an initial position, ac .

463. Two dihedral angles are *equal* when they can be placed so that their faces coincide.

464. A *right dihedral angle* has its plane angle a right angle, and its faces are said to be *perpendicular* to each other. In the same way, dihedral angles are *acute* or *obtuse*, and pairs of dihedral angles are *adjacent*, *complementary*, *supplementary*, *alternate*, *corresponding*, *vertical*, etc., according as their plane angles are acute, etc.

PROPOSITION XV. THEOREM.

465. *Two dihedral angles are equal if their plane angles are equal.*



Given: Two dihedral angles, $CABD$, $C'A'B'D'$, having equal plane angles, CAD , $C'A'D'$;

To Prove: Dihedral angle $CABD =$ dihedral angle $C'A'B'D'$.

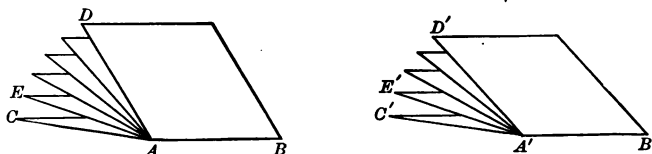
Apply $C'A'B'D'$ to $CABD$ so that $\angle C'A'D' \neq \angle CAD$.

Then the planes of these angles will coincide (423), and $A'B'$, AB , will coincide, both being perpendicular to the same plane at the same point (431); hence the planes $B'C'$ and BC , $B'D'$ and BD , will coincide (423);

\therefore dihedral $\angle CABD =$ dihedral $\angle C'A'B'D'$. Q.E.D. (463)

PROPOSITION XVI. THEOREM.

466. *Dihedral angles are to each other as their plane angles.*



Given: Two dihedral angles, $CABD$, $C'A'B'D'$, and their respective plane angles, CAD , $C'A'D'$;

To Prove: Dihedral angle $CABD$: dihedral angle $C'A'B'D' =$ angle CAD : angle $C'A'D'$.

1°. When $\angle CAD$ and $C'A'D'$ are commensurable.

Let $\angle CAE$ be a common measure of these angles, so that $\angle CAE$ is contained 5 times in CAD and 4 times in $C'A'D'$.

Draw lines $AE, A'E'$, etc., dividing $\angle CAD, C'A'D'$, into 5 and 4 equal parts respectively, and through these lines and the edges $AB, A'B'$, pass planes so as to divide the dihedral $\angle CABD, C'A'B'D'$, into 5 and 4 equal dihedral \angle s respectively.

Since $\angle CAD : \angle C'A'D' = 5 : 4$, (Hyp.)

and dihed. $\angle CABD : \text{dihed. } \angle C'A'B'D' = 5 : 4$, (Const.)

dihed. $\angle CABD : \text{dihed. } \angle C'A'B'D' = \angle CAD : \angle C'A'D'$. (232''')

2°. When $\angle CAD$ and $C'A'D'$ are incommensurable, we can prove by the method of limits, as in (260''), that always

dihed. $\angle CABD : \text{dihed. } \angle C'A'B'D' = \angle CAD : \angle C'A'D'$. Q.E.D.

467. COR. *Vertical dihedral angles are equal.*

For they have the ratio of their equal vertical plane angles.

468. SCHOLIUM. In like manner may be established the following properties of dihedral angles by means of the corresponding properties of plane angles.

(1) *All right dihedral angles are equal.*

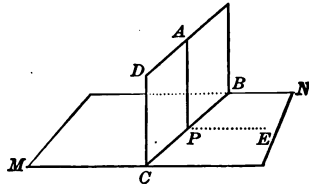
(2) *Two adjacent dihedral angles, formed by one plane meeting another, are supplementary; i.e., are equal to two right dihedral angles.*

(3) *Of the dihedral angles formed by a plane intersecting parallel planes, the alternate and corresponding angles are equal, and the interior angles on the same side of the transverse plane are supplementary.*

(4) *Dihedral angles having their faces mutually parallel, or, if their edges are parallel, respectively perpendicular to each other, are either equal or supplementary.*

PROPOSITION XVII. THEOREM.

469. *If a straight line is perpendicular to a given plane, every plane passed through that line is perpendicular to the given plane.*



Given: AP perpendicular to plane MN , and BD , a plane passing through AP ;

To Prove: Plane BD is perpendicular to plane MN .

Draw $PE \perp$ to BC , the intersection of BD and MN .

Since AP is \perp to plane MN , (Hyp.)

AP is \perp to BC and PE ; (427)

$\therefore APE$, a right angle,

is the plane angle that measures the dihedral angle formed by the planes intersecting in BC ; (462)

\therefore plane BD is \perp to plane MN . (464)

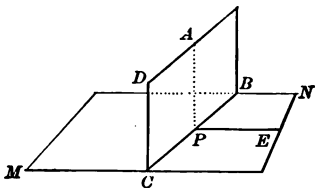
470. COR. *A plane perpendicular to the edge of a dihedral angle is perpendicular to each of its faces.*

EXERCISE 651. In the diagram for Prop. XIII., if from any point D in $\triangle ABC$, DD' be drawn parallel to AA' to meet $\triangle A'B'C'$ in D' , and D be joined with A and C , D' with A' and C' , the triangles DAC and $D'A'C'$ are equal.

652. If planes are passed through the sides of a triangle perpendicular to its plane, of the inner dihedral angles formed, no two are equal, two are equal, or all are equal, according as the triangle is scalene, isosceles, or equilateral.

PROPOSITION XVIII. THEOREM.

471. *If two planes are perpendicular to each other, any line in the one plane perpendicular to their intersection is perpendicular to the other.*



Given: In plane MN , perpendicular to plane BD , EP perpendicular to BC , the intersection of MN and BD ;

To Prove: EP is perpendicular to plane BD .

From P draw, in plane BD , $PA \perp$ to BC .

Since EP , PA , are each \perp to BC at P , (Hyp. and Const.)

$\angle APE$ is the plane angle that measures the dihedral angle contained by planes BD , MN , intersecting in BC . (462)

But plane BD is \perp to plane MN ; (Hyp.)

$\therefore \angle APE$ is a rt. \angle ;

$\therefore EP$ is \perp to plane BD , Q.E.D. (428)

(since it is \perp to BC and to PA in that plane.)

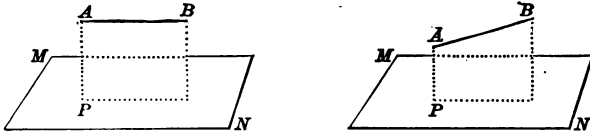
472. COR. 1. *If two planes, BD , MN , are perpendicular to each other, a straight line PE drawn at any point P of their intersection, so as to be perpendicular to one of the planes, as BD , will lie in the other plane MN .*

For in the plane of AP and PE , only one perpendicular can be drawn to AP at P (431).

473. COR. 2. *If two planes are perpendicular to each other, a perpendicular from any point of the one plane to the other must lie in the first plane.*

PROPOSITION XIX. THEOREM.

474. *Through any straight line not perpendicular to a plane, one plane, and only one, can be passed perpendicular to that plane.*



Given: AB , any straight line not perpendicular to plane MN ;

To Prove: Only one plane perpendicular to MN can be passed through AB .

Through A draw $AP \perp$ to plane MN , (429)

and pass through AB and AP a plane, BP .

Since AP is \perp to plane MN , (Const.)

plane BP is \perp to plane MN . (469)

But any plane passing through AB so as to be perpendicular to MN , must contain the perpendicular AP ; (472)

\therefore there can be but one such plane. Q.E.D. (423)

EXERCISE 653. In the diagram for Prop. XIV., show that $FG : FE = AC \cdot DF : BD \cdot AF$.

654. In the diagram for Prop. XVII., if in plane DB , GG' be drawn parallel to CB , and through GG' planes be passed intersecting plane MN in HH' and KK' , show that HH' is parallel to KK' .

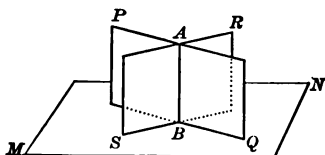
655. Show that if HH' and KK' are on opposite sides of CB , the dihedral angles formed by the plane GH' and GK' with MN , are or are not equal according as CB is or is not equidistant from HH' and KK' .

656. Show that, if BC is equidistant from HH' and KK' , plane DB bisects the dihedral angle $HGG'K'$.

657. If HH' and KK' are on the same side of CB , show that the dihedral angle formed by GH' with MN is the difference of that formed by GK' with MN and that formed by GH' with GK' .

PROPOSITION XX. THEOREM.

475. *If two intersecting planes are each perpendicular to a third plane, their intersection is also perpendicular to the third plane.*



Given: AB , the intersection of two planes, PQ , RS , each perpendicular to plane MN ;

To Prove: AB is perpendicular to plane MN .

At B erect a perpendicular BA to plane MN . (429)

Then BA lies in each of the planes PQ , RS ; (472)

$\therefore BA$ must coincide with the intersection of PQ , RS ;

$\therefore AB$ is \perp to MN .

Q.E.D.

476. COR. 1. *A plane perpendicular to each of two intersecting planes is perpendicular to their intersection.*

477. COR. 2. *If a plane is perpendicular to two planes perpendicular to each other, the intersection of any two of these planes is perpendicular to the third plane, and each intersection is perpendicular to the other two.*

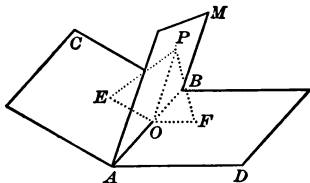
EXERCISE 658. Prove Cor. 2 of Prop. XX. by means of the diagram for Prop. XX., assuming that plane PQ is perpendicular to plane RS .

659. In the same diagram, if $\angle SBQ$ were 140° , what would be the ratio of dihedral angle $SBAQ$ to dihedral angle $SBAP$?

660. In the diagram for Prop. XXI., show that the points P , E , O , F , are concyclic.

PROPOSITION XXI. THEOREM.

478. *Every point in the plane that bisects a dihedral angle is equidistant from the faces of that angle.*



Given: P , a point in plane MA , bisecting dihedral angle $CABD$;

To Prove: P is equidistant from AC and BD .

From P draw $PE \perp$ to AC , and $PF \perp$ to BD ; (429)
 through PE and PF pass a plane intersecting AC in OE ,
 BD in OF , and therefore AM in OP .

Since PE is \perp to AC , and PF to BD , (Const.)

plane PEF is \perp to AB ; (476)

$\therefore POE$ and POF are the plane angles that measure the
 equal dihedral $\angle MABC, MABD$; (462)

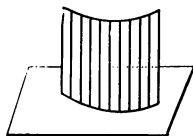
$\therefore \angle POE = \angle POF$;

\therefore rt. $\triangle PEO =$ rt. $\triangle PFO$; (73)

$\therefore PE = PF.$ Q.E.D.

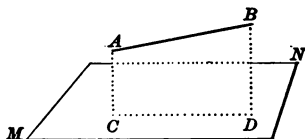
479. DEFINITION. The *projection of a point on a plane* is the foot of the perpendicular from the point to the plane.

480. DEFINITION. The *projection of a line on a plane* is the locus of the projections on that plane of all the points in the line.



PROPOSITION XXII. THEOREM.

481. *The projection of a straight line on a plane is a straight line.*



Given: A straight line AB and a plane MN ;

To Prove: The projection of AB upon MN is a straight line.

Through AB pass a plane $AD \perp$ to plane MN , which it intersects in CD .

Since AD is \perp to MN , (Const.)

AD contains all the perpendiculars from points in AB upon MN ; (473)

\therefore the feet of all these perpendiculars must meet MN in CD .

But CD is a straight line; (426)

\therefore the projection of AB on MN is a straight line. Q.E.D.

EXERCISE 661. In the diagram for Prop. XXI., how many degrees must there be in the plane angle of dihedral angle $CABD$, that FO may be equal to FP ?

662. In the same diagram, how many degrees are there in that plane angle if FP is equal to the line joining FE ?

663. Prove that if a line is equal to its projection on a given plane, it is parallel to the plane.

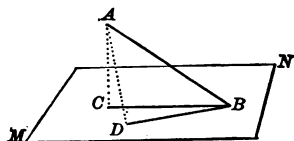
664. Prove that parallel lines have their projections on the same plane in lines that are coincident or parallel.

665. In the diagram for Prop. XXII., show that if BA be produced to meet MN in E , E is a point in DC produced.

666. In the diagram for the preceding exercise, if $BD : AC = m : n$, what is the ratio of CE to CD ?

PROPOSITION XXIII. THEOREM.

482. *The acute angle formed by a straight line with its own projection on a plane, is the least angle it makes with any line in that plane.*



Given: Angle ABC , formed by AB with its projection BC on plane MN ;

To Prove: Angle ABC is less than angle ABD , formed by AB with any other line in MN than BC .

Lay off $BD = BC$, and join AD .

In $\triangle ABC, ABD$, $AB = AB$, $BC = BD$, (Const.)

but $AC < AD$, (432)

(since AC is \perp to plane MN ;) (Hyp.)

$\therefore \angle ABC < \angle ABD$. Q.E.D. (91)

483. DEFINITION. The acute angle formed by a straight line with its own projection on a plane is called the *inclination of the line to the plane*, or the *angle of the line and plane*.

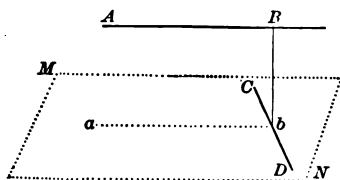
EXERCISE 667. Show that lines having their projections on the same plane, coincident or parallel, are not necessarily themselves parallel.

668. A line meets a plane obliquely; with what line in the plane does it make the greatest angle?

669. A line has an inclination of 42° to each of two intersecting planes; how many degrees are there in the plane angle of the dihedral angle formed by the planes?

PROPOSITION XXIV. THEOREM.

484. *A common perpendicular can be drawn to any two given straight lines not in the same plane.*



Given: AB and CD , two straight lines not in the same plane ;

To Prove: A common perpendicular can be drawn to AB and CD .

Through CD pass a plane MN that is \parallel to AB , (454)

and let ab be the projection of AB upon MN .

Since ab is \parallel to AB , (449)

ab is not \parallel to CD ,

(since AB and CD cannot be parallel;) (Hyp.)

$\therefore ab$ will meet CD , say in b .

At b draw $bb' \perp$ to ab in the projecting plane of AB .

Since AB is \parallel to ab in the same plane,

bb' is \perp to AB ; (107)

and since bb' is \perp to MN , (471)

bb' is \perp to CD also. Q.E.D. (427)

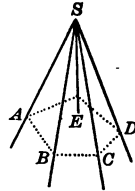
485. COR. 1. *Only one perpendicular can be drawn common to two straight lines not in the same plane.*

For if there could be two such perpendiculars, then, as can easily be shown, there could be two perpendiculars drawn in the same plane at the same point in a straight line, which is impossible (41).

486. COR. 2. *The common perpendicular is the shortest distance between two straight lines not in the same plane.*

POLYHEDRAL ANGLES.

487. A *polyhedral* or *solid angle* is the angle formed by three or more planes meeting in a common point. The point in which the planes meet is called the *vertex*; the intersections of the planes, the *edges*; and the portions of the planes bounded by the edges, the *faces* of the angle. Thus, in the polyhedral angle $S-ABCDE$, S is the vertex; SA , SB , etc., are the edges; and ASB , BSC , etc., are faces or *face angles*. It is to be noted that, in a polyhedral angle, every two adjacent edges form a face angle; and every two adjacent faces, a dihedral angle. It is also to be noted that the faces and edges of a polyhedral angle may be supposed to extend indefinitely. As a convenience in demonstration, however, portions of the faces and edges may be represented as cut off by a plane. The section formed by the intersection of the plane with the faces is a polygon, sometimes called the *base* of the polyhedral angle.



488. A polyhedral angle is *convex* if any section made by a plane cutting all its faces is a convex polygon; as $ABCDE$. It is to be understood that the polyhedral angles about to be treated of are convex.

489. A polyhedral angle is *trihedral*, *tetrahedral*, etc., according as it has *three*, *four*, etc., faces.

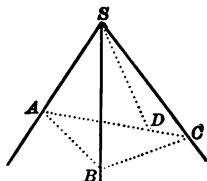
490. A trihedral angle is *rectangular*, *birectangular*, or *trirectangular*, according as it has one, two, or three right dihedral angles. The ceiling and walls of a room form trirectangular angles.

EXERCISE 670. In the diagram for Prop. XX., how many trihedral angles are represented, with what common vertex?

671. In the same diagram, if SBQ is a right angle, of what class, according to Art. 490, is each of the four trihedral angles?

PROPOSITION XXV. THEOREM.

491. In a trihedral angle, the sum of any two of the face angles is greater than the third face angle.



Given: In trihedral angle $S-ABC$, face angle CSA greater than angle ASB or angle BSC ;

To Prove: Angle ASB + angle BSC is greater than angle CSA .

In face CSA make $\angle ASD = \angle ASB$; (203)

through any point D of SD draw ADC in plane CSA ;

take $SB = SD$, and join AB, BC .

Since $SB = SD$, $SA = SA$, and $\angle ASB = \angle ASD$, (Const.)

$\triangle ASB = \triangle ASD$, and $AB = AD$. (66, 70)

In $\triangle ABC$, $AB + BC > AC$; (88)

$\therefore BC > (AC - AD)$ or DC . (Ax. 5)

In $\triangle BSC$, since $SB = SD$, $SC = SC$,

but $BC > DC$,

$\angle BSC > \angle DSC$; (91)

$\therefore \angle ASB + \angle BSC > \angle ASD + \angle DSC$; (Ax. 4)

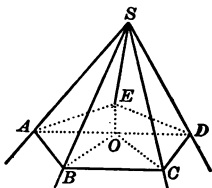
i.e., $\angle ASB + \angle BSC > \angle CSA$. Q.E.D.

EXERCISE 672. A plane can be passed perpendicular to only one edge or to two faces of a polyhedral angle.

673. If three lines in space are parallel, or meet in a common point, how many planes may they define, taken two and two?

PROPOSITION XXVI. THEOREM.

492. *The sum of the face angles of any convex polyhedral angle is less than four right angles.*



Given: ASB, BSC, CSD , etc., face angles of a polyhedral angle $S-ABCDE$;

To Prove: The sum of the angles ASB, BSC, CSD , etc., is less than four right angles.

Pass a plane so as to cut the edges in A, B, C, D, E , and the faces in AB, BC, CD, DE, EA ;

then $ABCDE$ is a convex polygon. (Hyp.)

Taking any point O in $ABCDE$, join OA, OB, OC, OD, OE .

As the number of triangles having their vertices at S is equal to the number of triangles having their vertices at O ,

(they having the same bases AB, BC , etc.,)

the sum of the interior \angle s of the one set is equal to that of the other set.

But in the trihedral \angle formed at A, B, C , etc.,

$$\left. \begin{aligned} \angle SBA + \angle SBC &> \angle ABC, \\ \text{and } \angle SCB + \angle SCD &> \angle BCD, \text{ etc.;} \end{aligned} \right\} \quad (491)$$

\therefore the sum of the \angle s at the bases of the triangles whose vertices are at S , is greater than that of the triangles whose vertices are at O ;

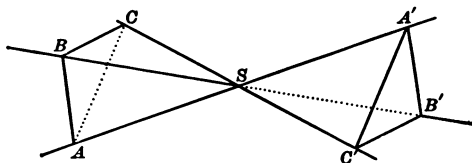
\therefore the sum of the \angle s at $S <$ the sum of the \angle s at O . (Ax. 5.)

But the sum of the \angle s at $O =$ four rt. \angle s;

\therefore the sum of the \angle s at $S <$ four rt. \angle s. Q.E.D.

PROPOSITION XXVII. THEOREM.

493. *If the edges of a trihedral angle be produced through the vertex, they will form the edges of a second trihedral angle, called the **symmetrical trihedral** of the first, with its face and dihedral angles respectively equal to those of the first, but arranged in reverse order.*



Given: Trihedral angle $S-A'B'C'$, formed by producing the edges of trihedral angle $S-ABC$;

To Prove: The face angles and dihedral angles of $S-A'B'C'$ are respectively equal to those of $S-ABC$, but in reverse order.

1°. The face $\angle A'SB'$, ASB , etc., are respectively equal, (50)

(being vertical \angle in planes determined by AA' , BB' , etc.)

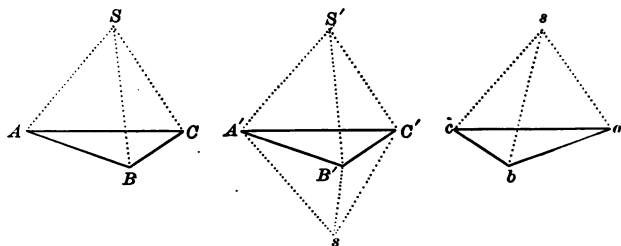
2°. The dihedral $\angle SA'$, SA , SB' , SB , etc., are respectively equal, (467)
(being vertical dihedral angles.)

3°. The angles of both kinds are in reverse order.

For to meet the faces of the first trihedral in the order ASB , BSC , CSA , we must turn from right to left,* while to meet the faces of the second trihedral in the order $A'SB'$, $B'SC'$, $C'SA'$, we must turn from left to right.

* In order to see this, look at each trihedral having the vertex above.

The matter, which is of some importance, may be made clearer by the following illustration. Let ABC , $A'B'C'$, abc , be three triangles having their sides and angles respectively equal; ABC , $A'B'C'$, having theirs in the same order, but abc in reverse order. It is obvious that ABC can be made to slide over and coincide with $A'B'C'$ without reversing, while abc must be taken up and reversed in order to apply it to $A'B'C'$, either from above or below, so as to make them coincide.



If, now, we conceive planes to be passed through the sides of the triangles and through points S , S' , s , similarly situated with respect to their vertices, it is evident that the trihedral angles thus formed will have their face angles and dihedral angles respectively equal, but those of $s-abc$ in reverse order from those of $S'-A'B'C'$ and $S-ABC$. Hence, if $\triangle ABC$ be made to coincide with $\triangle A'B'C'$, $S-ABC$ will coincide with $S'-A'B'C'$; since S will then coincide with S' . But if abc be made to coincide with $A'B'C'$, s will be on that side of $A'B'C'$ which is remote from S' , as in the figure.

494. Two polyhedral angles are *equal* and can be made to coincide if their face angles and dihedral angles are respectively equal and arranged in the same order; if these parts are equal but not arranged in the same order, the polyhedrals are said to be *symmetrical*.

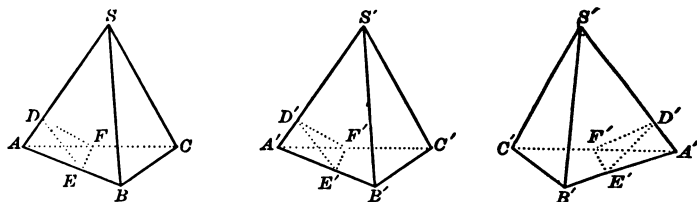
EXERCISE 674. If four lines in space are parallel, or meet in a common point, how many planes may they define, taken two and two?

675. A line parallel to two intersecting planes is parallel to their intersection.

676. If two unequal similar triangles not in the same plane have their sides respectively parallel, the lines joining their homologous vertices will, if produced, meet in one point.

PROPOSITION XXVIII. THEOREM.

495. *Trihedral angles that have their face angles respectively equal are equal or symmetrical.*



Given: Two trihedral triangles, $S-ABC$, $S'-A'B'C'$, having ASB equal to $A'S'B'$, BSC equal to $B'S'C'$, CSA equal to $C'S'A'$;

To Prove: $S-ABC$ and $S'-A'B'C'$ are equal or symmetrical.

On the six edges lay off $SA = S'A'$, $SB = S'B'$, $SC = S'C'$, and join AB , AC , BC , $A'B'$, $A'C'$, $B'C'$.

Since $\triangle SAB$, SBC , $SCA = \triangle S'A'B'$, $S'B'C'$, $S'C'A'$, respectively,

(66)

AB , AC , $BC = A'B'$, $A'C'$, $B'C'$, respectively;

$\therefore \triangle ABC = \triangle A'B'C'$ and $\angle BAC = \angle B'A'C'$. (69)

At any point D in SA draw DE , DF , each \perp to SA , in the faces ASB , ASC respectively. These lines will meet AB and AC respectively,

(114)

(since the $\angle SAB$, SAC , are acute, being base \angle of isos. \triangle .)

Let them meet AB , AC , in E and F , respectively, and join EF .

On $S'A'$ lay off $A'D' = AD$, and construct $D'E'F'$ as DEF was constructed. Then since

$AD = A'D'$ (Const.) and $\angle DAE = \angle D'A'E'$,

rt. $\triangle ADE =$ rt. $\triangle A'D'E'$; (63)

$\therefore AE = A'E'$ and $DE = D'E'$.

In like manner it may be shown that

$$AF = A'F' \text{ and } DF = D'F'.$$

Since $AE = A'E'$, $AF = A'F'$, and $\angle BAC = \angle B'A'C'$, (Above)

$$\triangle AEF = \triangle A'E'F', \text{ and } EF = E'F'. \quad (66)$$

Since $DE = D'E'$, $DF = D'F'$, and $EF = E'F'$,

$$\triangle DEF = \triangle D'E'F', \text{ and } \angle FDE = \angle F'D'E'; \quad (69)$$

$$\therefore \text{dihedral } \angle AS = \text{dihedral } \angle A'S', \quad (465)$$

(being measured by equal $\angle FDE, F'D'E'$.)

In the same way it may be proved that

dihed. $\angle BS = \text{dihed. } \angle B'S'$, and dihed. $\angle CS = \text{dihed. } \angle C'S'$.

Hence the trihedrals $S-ABC$, $S'-A'B'C'$, are equal or symmetrical according as the equal parts are or are not arranged in the same order. Q.E.D. (494)

496. COR. 1. *If two trihedral angles have the three face angles of the one respectively equal to the three face angles of the other, the dihedral angles of the one are respectively equal to the dihedral angles of the other.*

497. COR. 2. *An isosceles trihedral angle, that is, one having two of its face angles equal, is equal to its symmetrical trihedral.*

For if in $S-ABC$, we have $\angle ASB = \angle BSC$, then in $S'-A'B'C'$ we shall have $\angle A'S'B' = \angle B'S'C'$; and the $\angle CSA$, $C'S'A'$ will have equal faces on each side of them; then also the dihedral angles will be similarly arranged, and the trihedrals will be equal (495).

EXERCISE 677. If a line makes an acute angle with a plane, every plane with which it makes the same angle is parallel to the first.

678. Parallel lines that intersect the same plane make equal angles with it.

679. If the projections of any number of points upon a plane lie in one straight line, these points are in one plane. What plane?

EXERCISES.

QUESTIONS.

680. What space concepts are determined in position by one point, by two, by three, respectively ?

681. To what theorem in Book I. does Prop. V. correspond ?

682. What is the locus of all the points in space that are equidistant from a given circumference ?

683. What is the locus of all the points in space that are equidistant from the vertices of a triangle ?

684. What is the locus of all the points in space that are equidistant from the mid points of the sides of a triangle ?

685. To what theorem in Book I. does Cor. 3 of Prop. VI. correspond ?

686. What is the locus of all the lines that are perpendicular to a given line at a given point ?

687. To what theorems in Book I. do Prop. VIII. and its first corollary correspond ?

688. What is the locus of all the lines that have a given line in a given plane as their projection ?

689. What is the locus of all the points in space that are at a given distance from a given plane ?

690. To what theorems in Book I. do Prop. XII., Cor., and Prop. XIII. correspond ?

691. To what theorem in Book IV. does Prop. XIV. correspond ?

692. To what theorem in Book I. does Prop. XIX. correspond ?

693. What is the locus of all the points in space that are equidistant from two intersecting planes ? From two parallel planes ?

THEOREMS.

694. If a line is perpendicular to one of two intersecting planes, its projection on the other plane is perpendicular to the intersection.

695. A plane parallel to two sides of a quadrilateral in space, — that is, a quadrilateral having its sides two and two in different planes, — divides the other two sides proportionally.

696. The mid points of the sides of a quadrilateral in space are the angular points of a parallelogram.

697. If the intersections of any number of planes are parallel, the perpendiculars drawn to these planes from the same point in space are in the same plane.

698. If a line is equally inclined to both faces of a dihedral angle, the points in which it meets the faces are equally distant from the edge of the dihedral.

699. If a line makes equal angles with three lines in the same plane, it is perpendicular to that plane.

700. If a plane be passed through one diagonal of a parallelogram, the perpendiculars to that plane from the extremities of the other diagonal are equal.

701. If from a point within a dihedral angle, perpendiculars are drawn to its faces, the angle contained by these perpendiculars is equal to the plane angle of the adjacent dihedral angle formed by producing one of the planes.

702. If three planes have a common intersection, perpendiculars to these planes from any point in the intersection are in the same plane.

703. In any trihedral angle, the three planes bisecting its dihedral angles intersect in the same line.

704. In any trihedral angle, the three planes passed through its edges perpendicular to the opposite faces, intersect in the same line.

705. In any trihedral angle, the three planes passed through the edges and the bisectors of the opposite face angles, intersect in the same line.

706. In any trihedral angle, the three planes passed perpendicularly through the bisectors of the face angles, intersect in the same line.

LOCI.

Find the loci of the points in space that respectively satisfy the following conditions :

707. Are equidistant from two given points.

708. Are equidistant from two given intersecting lines.

709. Have their distances from two given planes in a given ratio.

- 710. Are equidistant from the vertices of a given triangle.
- 711. Are equidistant from the sides of a given triangle.
- 712. Are equidistant from the vertices of a quadrilateral whose opposite angles are supplementary.
- 713. Are equidistant from the circumference of a given circle.
- 714. Are equidistant from three given planes.
- 715. Are equidistant from the edges of a given trihedral angle.
- 716. Are equidistant from two given planes and two given points in space.

PROBLEMS.

In the construction of the following problems, it is assumed that, besides the constructions of Plane Geometry, we are able: (1) to pass a plane through any given line and any point or line that can be in the same plane with it (422, 423, 424); (2) through any given point in or without a given plane, to draw a perpendicular to that plane.

717. Through a given line in a plane pass a plane making a given angle with that plane.

718. Through a given line without a given plane pass a plane making a given angle with that plane.

719. Through the edge of a given dihedral angle pass a plane bisecting that angle.

720. Through a given point without a given plane pass a plane parallel to that plane.

721. At a given distance from a given plane pass a plane parallel to that plane.

722. Through a given point pass a plane perpendicular to a given straight line.

723. Through the vertex of a given trihedral angle draw a line making equal angles with the edges.

724. In a given plane find a point such that the lines drawn to it from two given points without the plane, shall make equal angles with the plane. (Two cases.)

725. In a given plane find a point equidistant from three given points without the plane.

726. In a given straight line find a point equidistant from two given points not in the same plane as the line.

BOOK VIII.

POLYHEDRONS.



498. A *polyhedron* is a solid bounded by four or more polygons. The bounding polygons are called the *faces*; their intersections, the *edges*; and the intersections of the edges, the *vertices* of the polyhedron.

The least number of planes that can form a solid angle — the trihedral — is three. As the faces and edges extend indefinitely, the space within the angle is of indefinite extent, so that in order to cut off a definite portion of that space, a fourth plane must be passed intersecting the faces. Hence four is the least number of planes that can inclose a space.

499. A polyhedron of four faces is called a *tetrahedron*; one of six faces, a *hexahedron*; of eight faces, an *octahedron*; of twelve faces, a *dodecahedron*; of twenty faces, an *icosahedron*.

500. A polyhedron is *convex* when every section of it by a plane is a convex polygon. As none but convex polyhedrons are to be treated of in what follows, the term *polyhedron* will always signify *convex polyhedron*.

501. The *volume* of a polyhedron is its quantity as measured by the polyhedron taken as *unit of volume*, or is the numerical measure of that quantity.

In every-day life, volume is expressed by stating how many stated solid measures a given solid contains; as 356 cu. in. In abstract discussions, however, by volume is

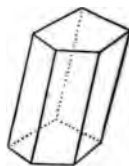
meant the numerical measure of a solid, or the ratio of a given solid to the solid unit.

502. *Similar polyhedrons* have the same form; *equivalent polyhedrons* have the same volume; *equal polyhedrons* have the same form and volume.

PRISMS.

503. A *prism* is a polyhedron two of whose faces are equal polygons having their homologous sides parallel, and whose other faces are parallelograms.

504. The *bases* of a prism are its equal parallel faces; the other faces are called *lateral faces*. The intersections of the bases and lateral faces are called *basal edges*; the intersections of the lateral faces are called *lateral edges*.

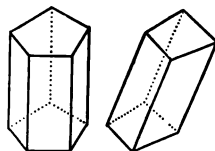


505. Prisms are *triangular*, *quadrangular*, *pentagonal*, etc., according as their bases are *triangles*, *quadrangles*, *pentagons*, etc.

506. A *right prism* has its lateral edges perpendicular to its bases; all other prisms are called *oblique prisms*.

507. A *regular prism* is a right prism whose bases are regular polygons.

COR. *The lateral faces of a regular prism are equal rectangles.*



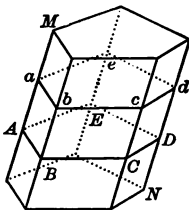
508. The *altitude* of a prism is the perpendicular distance between its bases.

COR. *Each lateral edge of a right prism is equal to the altitude; each lateral edge of an oblique prism is greater than the altitude.*

509. A *right section* of a prism is a section perpendicular to its lateral edges.

PROPOSITION I. THEOREM.

510. *The sections of a prism made by parallel planes are equal polygons.*



Given: A prism MN intersected by parallel planes ABD , abd ;

To Prove: $ABCDE$ is equal to $abcde$.

Since plane ABD is \parallel to plane abd , (Hyp.)

AB , BC , CD , etc., are \parallel to ab , bc , cd , etc., resp., (451)

and these lines are similarly directed;

$\therefore \angle A = \angle a$, $\angle B = \angle b$, $\angle C = \angle c$, etc.; (455)

$\therefore ABCDE$ and $abcde$ are mutually equiangular.

Since $AB = ab$, $BC = bc$, $CD = cd$, etc., (136)

$ABCDE$ and $abcde$ are mutually equilateral;

$\therefore ABCDE = abcde$. Q.E.D. (61)

511. COR. *Any section of a prism parallel to the base is equal to the base; also all right sections of a prism are equal.*

512. DEFINITION. *The lateral or convex surface of a prism is the sum of its lateral faces.*

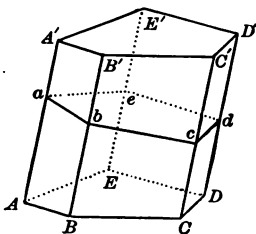
EXERCISE 727. In the diagram above, if the prism MN is pentagonal, what is the sum of the plane angles of the lateral dihedral angles?

728. Supposing, as above, that MN is pentagonal, how many faces, face angles, dihedral and trihedral angles, has MN ?

729. If the base have n sides, how many faces, etc., has MN ?

PROPOSITION II. THEOREM.

513. *The lateral surface of a prism is equivalent to the rectangle contained by a lateral edge and the perimeter of a right section of the prism.*



Given: A right section $abcde$, and AA' , a lateral edge of prism AD' ;

To Prove: The lateral surface of AD' is equivalent to rectangle $AA' \cdot (ab + bc + \text{etc.})$.

Since AD' is a prism, (Hyp.)

AB', BC', CD' , etc., are parallelograms. (503)

Since $abcde$ is a right section, (Hyp.)

ab, bc, cd , etc., are \perp to AA', BB', CC' , etc., resp.; (509)

$\therefore AB' \approx AA' \cdot ab$; $BC' \approx BB' \cdot bc$, etc.;

$\therefore AB' + BC' + CD'$, etc., $\approx AA' \cdot ab + BB' \cdot bc + CC' \cdot cd$, etc.; (Ax. 2)

\therefore lat. surf. of $AD' \approx \text{rect. } AA' \cdot (ab + bc + \text{etc.})$, Q.E.D.

(since $AA' = BB' = CC' = \text{etc.}$) (136)

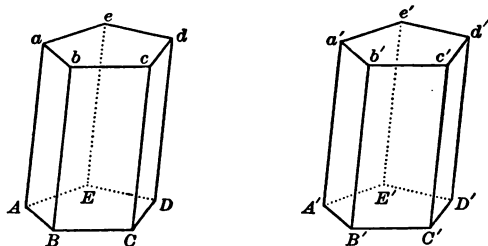
514. COR. *The lateral surface of a right prism is equivalent to the rectangle of its altitude and the perimeter of its base.*

SCHOLIUM. Prop. II. may also be expressed under the form: *The lateral area of a prism is equal to the product of a lateral edge and the perimeter of a right section.*

In the same way, in the corollary to Prop. II. and similar theorems, area may be substituted for surface, and product for rectangle, as explained in Art. 324.

PROPOSITION III. THEOREM.

515. *Two prisms are equal, if three faces including a trihedral angle of the one are respectively equal to three similarly arranged faces including a trihedral angle of the other.*



Given: In the prisms Ad , $A'd'$, the faces AD , Ab , Ae , respectively equal to the faces $A'D'$, $A'b'$, $A'e'$, and similarly arranged;

To Prove: Prism Ad is equal to prism $A'd'$.

Since $AD = A'D'$, $Ab = A'b'$, $Ae = A'e'$,
and they are similarly arranged, } (Hyp.)
trihedral $\angle A = \text{trihedral } \angle A'$. (495)

Hence, applying trihed. $\angle A$ to trihed. $\angle A'$, they will coincide;

then since $AD \neq A'D'$, $Ab \neq A'b'$, and $Ae \neq A'e'$,

edge $ab \neq \text{edge } a'b'$, and edge $ae \neq a'e'$;

\therefore base $ad \neq \text{base } a'd'$,

(being equal polygons having two sides coinciding;)

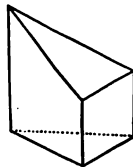
\therefore all the lateral edges will coincide,

(since their extremities coincide;)

\therefore the prisms coincide and are equal. Q.E.D. . (61)

516. DEFINITION. A *truncated prism* is a part of a prism cut off by a plane not parallel to the base.

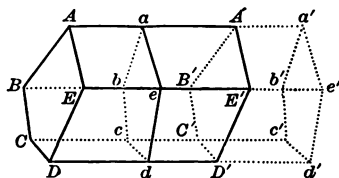
517. COR. 1. Two truncated prisms are equal, if the three faces including a trihedral angle of the one are respectively equal to the three faces including a trihedral angle of the other, and are similarly arranged.



518. COR. 2. Two right prisms are equal if they have equal bases and equal altitudes.

PROPOSITION IV. THEOREM.

519. An oblique prism is equivalent to a right prism having as base a right section, and as altitude a lateral edge, of the oblique prism.



Given: In oblique prism AD' , a right section, $abcde$, and a lateral edge, AA' ;

To Prove: AD' is equivalent to a right prism having as base $abcde$, and an altitude equal to AA' .

Produce AA' to a' , so that $aa' = AA'$; and through a' pass a plane \perp to aa' , and intersecting all the faces of the prism produced, thus forming a second right section, $a'b'c'd'e'$, parallel and equal to $abcde$.

The prism ad' thus formed is a right prism (506), whose base is the given right section, $abcde$, and whose altitude $aa' = AA'$ (Const.).

Now the figures $ABCDE-d$ and $A'B'C'D'E'-d'$ are equal truncated prisms (517); and, if to each of these we add the figure $abcde-D'$, we obtain (Ax. 2)

prism $AD' \approx$ rt. prism $abcde-d'$. Q.E.D.

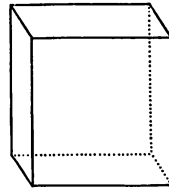
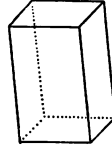
520. DEFINITION. A *parallelopiped* is a prism whose bases are parallelograms.

COR. Any two opposite faces of a parallelopiped are equal parallelograms.

521. DEFINITION. A *right parallelopiped* has its lateral edges \perp to its bases.

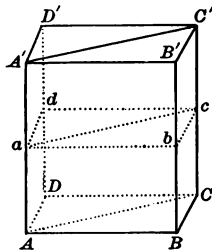
522. DEFINITION. A *rectangular parallelopiped* is a right parallelopiped whose bases are rectangles.

COR. Hence all its faces are rectangles.



PROPOSITION V. THEOREM.

523. The plane passed through two diagonally opposite edges of a parallelopiped divides it into two equivalent triangular prisms.



Given: A plane AC' through diagonally opposite edges of parallelopiped BD' ;

To Prove: Triangular prism $ABC-B'$ is equivalent to triangular prism $ADC-D'$.

Pass a plane \perp to AA' so as to form a rt. section $abcd$ of the parallelepiped, and intersecting AC' in ac .

Since ac is a diagonal of par'm $abcd$,

$$\Delta abc = \Delta adc. \quad (140)$$

$$\left. \begin{array}{l} \text{Now } ABC-B' \approx \text{a rt. prism having for base} \\ \quad \Delta abc, \text{ and alt.} = BB'; \\ \text{and } ADC-D' \approx \text{a rt. prism having for base} \\ \quad \Delta adc, \text{ and alt.} = BB'; \end{array} \right\} \quad (519)$$

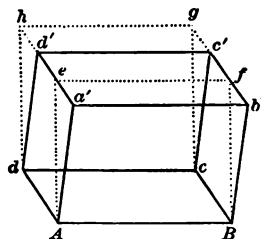
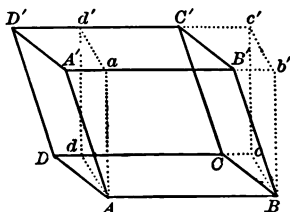
\therefore prism $ABC-B' \approx$ prism $ADC-D'$, . Q.E.D. (Ax. 1)

(since they are equivalent to equal rt. prisms.)

524. COR. *Any triangular prism is equivalent to one half the parallelepiped having the same altitude and a base twice as great.*

PROPOSITION VI. THEOREM.

525. *Any parallelepiped is equivalent to a rectangular parallelepiped having the same altitude and an equivalent base.*



Given: A parallelepiped $ABCD-C'$, having an altitude H ;

To Prove: $ABCD-C'$ is equivalent to a rectangular parallelepiped having a base equivalent to $ABCD$, and altitude equal to H .

Through A and B pass planes each \perp to AB .

By the intersections of these planes with the faces, or the faces produced, a new parallelopiped, $ABcd-c'$, will be formed, of the same altitude as the given figure, and having an equivalent base (Const., 328).

Now $ABcd-c' \approx ABCD-C'$ (519); for taking AD' as the base of the latter, then $ABcd-c'$ is a rt. prism whose base, Ad' , is a rt. section of the given prism, and whose altitude is AB , a lateral edge of the same.

Again (Fig. 2), through the edges AB and dc pass planes Af , dg , \perp to Ac , the base.

By the intersection of these planes with the faces, produced if necessary, of $ABcd-c'$, a rectangular parallelopiped, $ABcd-g$, will be formed, having the same base and altitude as $ABcd-c'$, or $ABb'a'-c'$.

Now, by regarding Ab' as base of $ABb'a'-c'$, and Af as a right section, it is seen that

$$ABcd-g \approx ABcd-c' \text{ (519). Hence (Ax. 1)}$$

$$ABCD-C' \approx ABcd-g, \text{ a rect. p'ped, etc.,} \quad \text{Q.E.D.}$$

(since each is equivalent to $ABcd-c'$.)

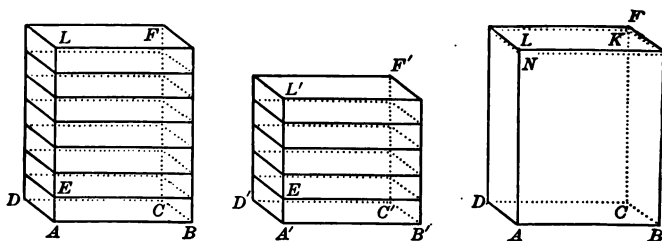
To obviate a frequent cause of difficulty to the student, it may be well to remark that the necessity of the double construction originates in the fact that the given solid $ABCD-C'$ being *any* parallelopiped, we have to assume the possibility that both pairs of faces, AB' and DC' , AD' and BC' , are oblique to the base AC . If, in the given solid, we suppose AB' perpendicular to the base, then $ABcd-c'$, the parallelopiped first constructed, will be rectangular, and no further construction be necessary.

EXERCISE 730. In the diagram for Prop. VI., left-hand figure, show that the solid $AA'D'D-d'$ is equivalent to the solid $BB'C'C-c'$.

731. In $AA'D'D-d'$, what kinds of polyhedral angles are those having their vertices at A and D , respectively?

PROPOSITION VII. THEOREM.

526. *Rectangular parallelepipeds with equal bases are to each other as their altitudes.*



Given: Two rectangular parallelepipeds, AF , $A'F'$, with equal bases, and altitudes AL , $A'L'$;

To Prove: $\text{p'ped } AF : \text{p'ped } A'F' = AL : A'L'$.

1°. When AL and $A'L'$ are commensurable.

Let $A'E$ be a common measure of AL and $A'L'$, so that

$A'E$ can be laid off 7 times on AL and 5 times on $A'L'$.

Through the points of division pass planes \perp to the edges.

Having equal bases (510) and equal altitudes, (Const.)

the p'peds thus formed are equal. (518)

Since $AL : A'L' = 7 : 5$, (Hyp.)

and $\text{p'ped } AF : \text{p'ped } A'F' = 7 : 5$, (Const.)

$\text{p'ped } AF : \text{p'ped } A'F' = AL : A'L'$. Q.E.D. (P. 232''')

2°. When AL and $A'L'$ are incommensurable.

Suppose $A'L'$ divided into any number of equal parts, n , and that AL contains m such parts with a remainder NL . Through N pass a plane perpendicular to the edges and cutting off the parallelepiped AK .

Since AN and $A'L'$ are commensurable, (Const.)

$$\frac{AN}{A'L'} = \frac{m}{n} = \frac{\text{p'ped } AK}{\text{p'ped } A'F'}; \quad (1^\circ)$$

$$\therefore \frac{AL}{A'L'} = \frac{m}{n} + \frac{x}{n} \text{ and } \frac{AF}{A'F'} = \frac{m}{n} + \frac{x'}{n},$$

since AL and AK are slightly greater than AN and AK resp.

Now when n is taken indefinitely great, $\frac{x}{n}$ and $\frac{x'}{n}$ become indefinitely small;

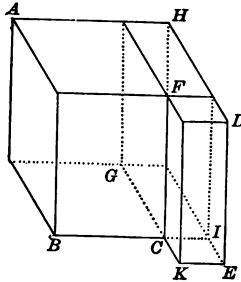
$$\therefore \frac{\text{p'ped } AF}{\text{p'ped } A'F'} = \frac{AL}{A'L'}, \quad \text{Q.E.D. (255)}$$

(being the limits of variables always equal.)

527. COR. *If two rectangular parallelopeds have two dimensions in common, they are to each other as their third dimensions.*

PROPOSITION VIII. THEOREM.

528. *Rectangular parallelopeds having equal altitudes are to each other as their bases.*



Given: Two rectangular parallelopeds, CA , CD , having equal altitudes, and bases BG , CE ;

To Prove: P'ped CA : p'ped CD = base BG : base CE .

Place the parallelopipeds so that the edge CF may be common, and the right dihedral angles at CF vertical.

Produce the faces AG , BG , AF , DI , so as to meet, and form a third rectangular parallelopiped, CH .

Since CA , CH , have the same base, FG , and the altitudes BC , CI ,

$$CA : CH = BC : CI = BC \cdot CG : CI \cdot CG. \quad (526, 318)$$

Since CD , CH , have the same base, FI , and altitudes CK , CG ,

$$CD : CH = CK : CG = CI \cdot CK : CI \cdot CG; \quad (526, 318)$$

$$\therefore CA : CD = \text{rect. } BC \cdot CG : \text{rect. } CI \cdot CK; \quad (249)$$

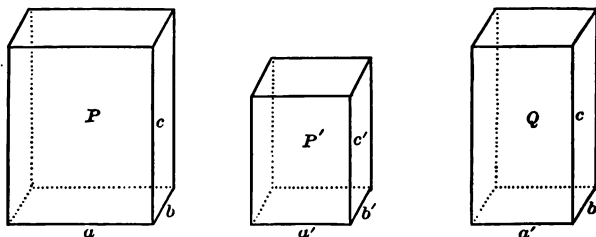
i.e., p'ped CA : p'ped CD = base BG : base CE . Q.E.D.

529. SCHOLIUM. The foregoing proposition may be expressed as follows :

If two rectangular parallelopipeds have one dimension in common, they are to each other as the products of their other two dimensions.

PROPOSITION IX. THEOREM.

530. *Rectangular parallelopipeds are to each other as the products of their three dimensions.*



Given : Two rectangular parallelopipeds, P , P' , with dimensions a , b , c , and a' , b' , c' , respectively;

To Prove : $P : P' = a \times b \times c : a' \times b' \times c'$.

Let Q be a third rectangular parallelopiped whose dimensions are a' , b , c .

Since P , Q , have two dimensions b , c , in common, (Hyp.)

$$P : Q = a : a'. \quad (527)$$

Since Q , P' , have the dimension a' in common, (Hyp.)

$$Q : P' = b \times c : b' \times c'. \quad (529)$$

$$\therefore P : P' = a \times b \times c : a' \times b' \times c'. \text{ Q.E.D. } \quad (242)$$

531. DEFINITION. A *cube* is a rectangular parallelopiped whose faces are all squares.

COR. *The edges of a cube are all equal.*

532. DEFINITION. The *unit of volume* is the cube whose edge is the linear unit, and whose base is, consequently, the unit of area.

533. COR. 1. *The volume of a rectangular parallelopiped is measured by the product of its three dimensions.*

For if a , b , c , be the dimensions of a rectangular parallelopiped P , and U be the unit cube, then (530),

$$P : U = abc : 1 \times 1 \times 1;$$

$$\therefore P = U \times abc.$$

This could be expressed at greater length as follows:
The number of unit cubes in any rectangular parallelopiped is equal to the number of units in the product of the numerical measures of its length, breadth, and thickness.

534. COR. 2. *The volume of a cube is measured by the cube of its edge.*

Thus if the edge of a cube be 7 linear units, the cube contains $7^3 = 343$ unit cubes; if the edge be a linear units, the cube contains a^3 times the unit cube.

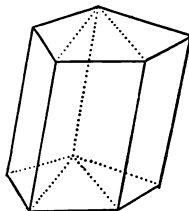
PROPOSITION X. THEOREM.

535. *The volume of any prism is measured by the product of its base and altitude.*

1°. Any parallelopiped is equivalent to a rectangular parallelopiped having the same altitude and an equivalent base (525); and the volume of the latter is measured by the product of its three dimensions,—that is, of its base and altitude (533); hence *the volume of any parallelopiped is measured by the product of its base and altitude.*

2°. Any triangular prism is equivalent to one half the parallelopiped having the same altitude and a base of twice the area (524); now, the volume of the latter being measured by the product of its base and altitude (1°), *the volume of the triangular prism is also measured by the product of its base and altitude.*

3°. By passing planes through its lateral edges, any prism can be divided into triangular prisms whose altitudes are the same as that of the given prism, and whose triangular bases together form the base of the given prism. As the volume of each of these triangular prisms is measured by the product of its base and altitude (2°), *the volume of any prism is measured by the product of its base and altitude.* Q.E.D.



536. COR. *Prisms having equivalent bases are to each other as their altitudes; prisms having equal altitudes are to each other as their bases; and prisms are to each other as the products of their bases and altitudes.*

EXERCISE 732. Two triangular prisms, *A* and *B*, have the same altitude. *A* has for base a right-isosceles triangle; *B*, for base an equilateral triangle of side equal to the hypotenuse of the base of *A*. Find the ratio of the volume of *A* to that of *B*.

733. Find the ratio of the lateral area of *A* to that of *B*.

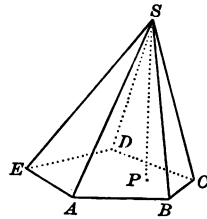
PYRAMIDS.

537. A *pyramid* is a polyhedron bounded by a polygon called the *base*, and by triangular planes meeting in a common point called the *vertex*. A plane intersecting the faces of any polyhedral angle cuts off a pyramid.

The terms *lateral face*, *lateral surface*, *lateral edge*, *basal edge*, are defined as for prisms (504).

538. The *altitude* of a pyramid is the perpendicular distance from its vertex to the base; as SP .

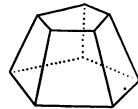
539. A *regular pyramid* has for base a regular polygon, and has its vertex in the perpendicular at the center of the base, which perpendicular is called the *axis* of the pyramid.



540. The *slant height* of a regular pyramid is the altitude of any lateral face.

541. A pyramid is *triangular*, *quadrangular*, *pentagonal*, etc., according as its base is a *triangle*, *quadrangle*, *pentagon*, etc. In the triangular pyramid, or tetrahedron (499), any one of the faces may be regarded as the base.

542. A *truncated pyramid* is the portion of a pyramid included between the base and a plane that intersects all the lateral faces.



543. A *frustum* of a pyramid is a truncated pyramid in which the intersecting plane is parallel to the base. The base of the pyramid is called the *lower base* of the frustum; the parallel section, the *upper base*.

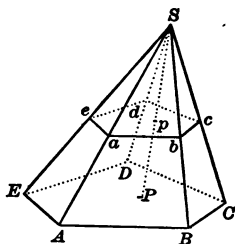
544. The *altitude* of a frustum is the perpendicular distance between its bases; the *slant height* of a frustum of a regular pyramid is the altitude of any lateral face.

PROPOSITION XI. THEOREM.

545. If a pyramid be cut by a plane parallel to the base:

1°. The edges and altitude will be divided proportionally.

2°. The section is a polygon similar to the base.



Given: A pyramid $S-ABD$, whose altitude SP is cut in p by a plane abd parallel to the base;

To Prove: 1°, $SA : Sa = SB : Sb = SP : Sp$, etc.;

2°, abd is similar to ABD .

1°. Suppose a plane passed through $S \parallel$ to ABD .

Since the edges and altitude are cut by \parallel planes,

(Hyp. and Const.)

$$SA : Sa = SB : Sb = SC : Sc = SP : Sp, \text{ etc. } \quad \text{Q.E.D.} \quad (459)$$

2°. Since plane abd is \parallel to plane ABD , (Hyp.)

$$ab \text{ is } \parallel \text{ to } AB, bc \text{ is } \parallel \text{ to } BC, cd \text{ is } \parallel \text{ to } CD, \text{ etc.,} \quad (451)$$

$$\text{and they are similarly directed,} \quad (115)$$

$$abd \text{ and } ABD \text{ are mutually equiangular.} \quad (455)$$

Since ab is \parallel to AB , and bc is \parallel to BC ,

$$\triangle Sab \text{ is sim. to } \triangle SAB, \text{ and } \triangle Sbc \text{ to } \triangle SBC; \quad (291)$$

$$\therefore ab : AB = Sb : SB, \text{ and } bc : BC = Sb : SB; \quad (284)$$

$$\therefore ab : AB = bc : BC.$$

In the same way we show that

$$bc : BC = cd : CD = de : DE, \text{ etc.};$$

$\therefore abcde$ is similar to $ABCDE$. Q.E.D. (284)

546. COR. 1. *The area of any section of a pyramid parallel to the base, is proportional to the square of its distance from the vertex.*

For parallel sections being similar to the base (545), their areas are proportional to the squares of their homologous sides (344). Thus

$$abd : ABD = \overline{ab}^2 : \overline{AB}^2 = \overline{sa}^2 : \overline{SA}^2 = \overline{sp}^2 : \overline{SP}^2.$$

547. COR. 2. *If two pyramids, $S-ABD$, $S'-A'B'D'$, having equal altitudes, SP , $S'P'$, are cut by planes parallel to their bases, and at equal distances, sp , $s'p'$, from their vertices, the sections, abd , $a'b'd'$, will be to each other as the bases.*

$$\left. \begin{array}{l} \text{For } abd : ABD = \overline{sp}^2 : \overline{SP}^2, \\ \text{and } a'b'd' : A'B'D' = \overline{s'p'}^2 : \overline{S'P'}^2; \end{array} \right\} \quad (546)$$

$$\text{but } sp = s'p' \text{ and } SP = S'P'; \quad (\text{Hyp.})$$

$$\therefore abd : ABD = a'b'd' : A'B'D'.$$

548. COR. 3. *If two pyramids have equal altitudes and equivalent bases, sections made by planes parallel to their bases, and at equal distances from the vertices, are equivalent.*

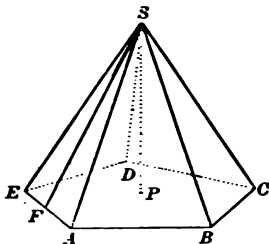
EXERCISE 734. Show that a plane perpendicular to the axis of a regular pyramid forms equal dihedral angles with all the faces of the pyramid.

735. In order that a plane intersecting the faces of a polyhedral angle may cut off a regular pyramid, what conditions must be fulfilled in regard to the form of the polyhedral and the inclination of the plane?

736. In the diagram for Prop. XI., if a plane be passed through the mid point of pP , parallel to the base, show that the perimeter of the section thus formed will be equal to half the sum of the perimeters of ABD and abd .

PROPOSITION XII. THEOREM.

549. *The lateral surface of a regular pyramid is equivalent to one half the rectangle contained by its perimeter and slant height.*



Given: A regular pyramid $S-ABD$, and SF its slant height;

To Prove: The lateral surface of $S-ABD$ is equivalent to

$$\frac{1}{2} \text{ rect. } SF \cdot (AB + BC + \dots EA).$$

Since ABD is a regular polygon, (Hyp.)

$$AB = BC = CD = DE = EA; \quad (370)$$

since A, B, C, D, E are equally distant from P , (539)

$$SA = SB = SC = SD = SE; \quad (437)$$

\therefore isos. $\triangle SAB =$ isos. $\triangle SBC =$ isos. $\triangle SCD$, etc. (69)

$$\text{But } \triangle SAE \approx \frac{1}{2} \text{ rect. } SF \cdot AE; \quad (331)$$

$$\therefore \triangle SAB + \triangle SBC + \dots \triangle SAE$$

$$\approx \frac{1}{2} SF \cdot (AB + BC + \dots AE); \quad (335)$$

$$\text{i.e., lat. surf. of } S-ABD \approx \frac{1}{2} SF \cdot \text{perimeter.} \quad \text{Q.E.D.}$$

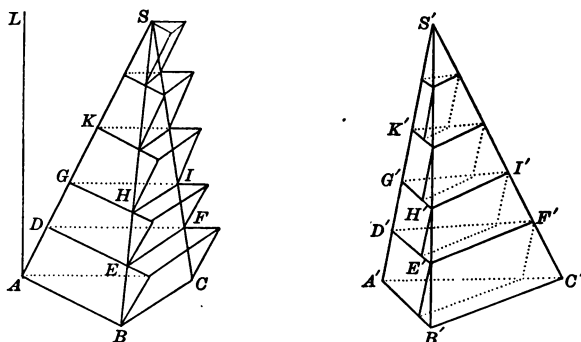
550. COR. 1. *The lateral surface of a frustum of a regular pyramid is equivalent to one half the rectangle contained by the slant height of the frustum and the sum of the perimeters of the bases.* (338)

For it is the sum of as many trapezoids as the base has sides, having for common altitude the slant height of the frustum. (544)

551. COR. 2. *The dihedral and trihedral angles at the base of a regular pyramid are all equal.*

PROPOSITION XIII. THEOREM.

552. *Triangular pyramids having equivalent bases and equal altitudes are equivalent.*



Given: Two triangular pyramids, $S-ABC$ and $S'-A'B'C'$, with equivalent bases, ABC and $A'B'C'$, and the same altitude AL ;

To Prove: $S-ABC$ is equivalent to $S'-A'B'C'$.

Place the pyramids so that they shall be in the same plane, and AL be their common altitude.

Divide AL into n equal parts, Aa , etc., and through the points of division pass planes parallel to the plane of the bases.

The corresponding sections thus formed are equivalent; (548)

that is, $DEF \approx D'E'F'$, $GHI \approx G'H'I'$, etc.

On the triangles ABC , DEF , etc., as lower bases, construct prisms $D-ABC$, $G-DEF$, etc., whose lateral edges are parallel to SA , and whose altitudes are each equal to Aa .

On the triangles $D'E'F'$, $G'H'I'$, etc., as upper bases, construct prisms $A'-D'E'F'$, $D'-G'H'I'$, etc., whose lateral edges are parallel to $S'A'$, and whose altitudes are each equal to Aa .

Now prism $G-DEF \approx$ prism $A'-D'E'F'$, (536)

because they have equivalent bases and the same altitude.

Similarly, $K-GHI \approx D'-G'H'I'$, etc.

That is, corresponding to each triangular prism constructed upon a section of $S-ABC$ as *lower base*, is an equivalent prism constructed upon the corresponding section of $S'-A'B'C'$ as *upper base*. Hence, the sum of all the prisms circumscribing $S-ABC$ differs from the sum of the prisms inscribed in $S'-A'B'C'$ by the prism $D-ABC$.

But the sum of all the prisms circumscribing $S-ABC$ is greater than that pyramid; and the sum of all the prisms inscribed in $S'-A'B'C'$ is less than $S'-A'B'C'$. Hence, these pyramids differ in volume by a volume less than prism $D-ABC$.

Now if n be taken indefinitely great, the altitude Aa becomes infinitesimal, and therefore the prism $D-ABC$ becomes infinitesimal. Hence, as they cannot differ by even an infinitesimal volume,

pyramid $S-ABC \approx$ pyramid $S'-A'B'C'$. Q.E.D.

EXERCISE 737. In the diagram for Prop. XII., if the axis is equal to the apothem of the base, what is the inclination of each face to the base?

738. In the same diagram, if the base is a regular pentagon, and $\angle SAB = 70^\circ$, what is the sum of the face angles?

739. In the same diagram, show that the mid point of SF is equidistant from S and P .

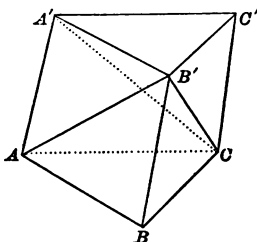
740. In a regular pyramid, the sum of the squares of the lateral edges is equivalent to one fourth of the sum of the squares of the basal edges, and n times the square of the slant height.

741. The perpendicular from the foot of the axis of a regular pyramid to the slant height is a mean proportional between the segments into which it divides it.

742. The axis and the slant height of a regular pyramid being given, find the apothem of the base.

PROPOSITION XIV. THEOREM.

553. *A triangular pyramid is one third of a triangular prism having the same base and altitude.*



Given: A triangular pyramid $B'-ABC$, and a triangular prism $ABC-C'$, on the same base.

To Prove: $B'-ABC$ is equivalent to $\frac{1}{3} ABC-C'$.

Take away the pyramid $B'-ABC$; there remains the quadrangular pyramid whose vertex is B' and whose base is the parallelogram AC' .

Through B' , A' , C , pass a plane. It will divide the pyramid $B'-AA'C'C$ into two triangular pyramids, which are equivalent (552) since the bases are halves of the parallelogram AC' , and they have the same altitude, — the perpendicular from B' upon the base AC' .

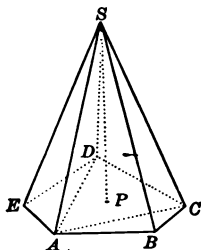
But pyramid $B'-A'C'C$, or $C-A'B'C'$, \approx pyramid $B'-ABC$ (552); for they have equal bases, $A'B'C'$, ABC (503), and the same altitude, namely, that of the prism. Hence

$$\begin{aligned} \text{pyr. } B'-ABC + \text{pyr. } B'-AA'C'C \\ + \text{pyr. } B'-A'C'C &\approx 3 \text{ pyr. } B'-ABC; \\ \text{i.e., prism } ABC-C' &\approx 3 \text{ pyr. } B'-ABC. \quad \text{Q.E.D.} \end{aligned}$$

554. *Cor. The volume of a triangular pyramid is measured by one third the product of its base and altitude.*

PROPOSITION XV. THEOREM.

555. *Any pyramid is one third of a prism having the same base and altitude.*



Given: Any pyramid $S-ABD$, with base ABD and altitude SP ;

To Prove: $S-ABD$ is equivalent to one third the prism on base ABD with altitude SP .

Through SA pass planes SAD , SAC , etc. These planes divide $S-ABD$ into triangular pyramids, whose bases make up the base ABD , and whose common altitude is SP .

Each of these triangular pyramids is one third of the triangular prism with altitude SP that could be constructed on the same base (553).

Hence the sum of the triangular pyramids that make up the given pyramid, is one third of the prism with altitude SP that could be constructed on the base ABD . Q.E.D.

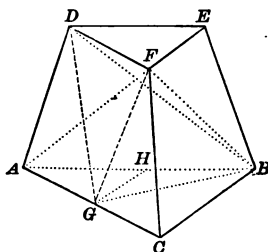
SCHOLIUM 1. The proposition could be expressed under the form: *The volume of any pyramid is measured by one third the product of its base and altitude.*

556. SCHOLIUM 2. The volume of any polyhedron may be found by dividing it into prisms or pyramids, and computing the sum of their volumes.

557. COR. *Pyramids of equivalent bases are as their altitudes; pyramids of equal altitudes are as their bases; any two pyramids are as the products of their bases and altitudes.*

PROPOSITION XVI. THEOREM.

558. *A frustum of a triangular pyramid is equivalent to the sum of three pyramids of the same altitude as the frustum, and whose bases are those of the frustum and a mean proportional between them.*



Given: A frustum $D-ABC$, with bases ABC , DEF ;

To Prove: $D-ABC$ is equivalent to three pyramids having the altitude of the frustum, and having as bases ABC , DEF , and a mean proportional between ABC and DEF .

Pass planes through the points F, A, B , and through F, D, B .

We thus divide the frustum into three triangular pyramids,

$F-ABC$, $B-DEF$, and $F-ADB$.

The first two of these evidently have the same altitude as the frustum, and have for bases ABC and DEF respectively. The pyramid $F-ADB$, we shall show, is equivalent to a triangular pyramid having the same altitude as the frustum, and a base that is a mean proportional between ABC and DEF .

In the plane DC , draw $FG \parallel$ to DA , and pass a plane through FGB .

$$FG \text{ is } \parallel \text{ to plane } ABED, \quad (448)$$

whence F and G are equally distant from that plane;

$$\therefore \text{pyr. } G-ADB \approx F-ADB. \quad (552)$$

If, again, we take D as the vertex, and AGB as the base of $G-ADB$, its altitude is the same as that of the frustum.

Draw $GH \parallel$ to CB . Then $\angle AGH = \angle DFE$,
and $\angle GAH = \angle FDE$. (455)

Since AF is a par'm, $AG = DF$;

$\therefore \triangle AHG = \triangle DEF$ (63), and $AH = DE$.

Since $\triangle ABC$ and ABG , $\triangle ABG$ and AHG , have equal alts.,

$$\left. \begin{array}{l} \triangle ABC : \triangle ABG = AC : AG = AC : DF, \\ \text{and } \triangle ABG : \triangle AHG = AB : AH = AB : DE. \end{array} \right\} \quad (333)$$

But $AC : DF = AB : DE$,

($\triangle ABC$ and DEF being similar.) (545")

$\therefore ABC : ABG = ABG : AHG = ABG : DEF$; (232''')

i.e., $\triangle ABG$ is a mean prop. between $\triangle ABC$ and DEF .

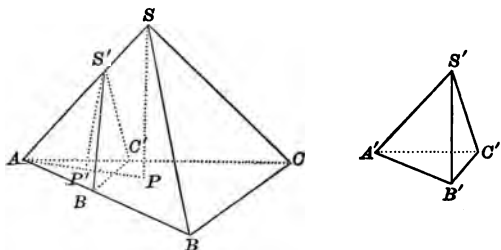
559. COR. 1. *The volume of a frustum of a triangular pyramid is measured by the product of one third its altitude into the sum of its bases and a mean proportional between them.*

560. COR. 2. *The volume of a frustum of any pyramid is measured by the product of one third its altitude into the sum of its bases and a mean proportional between them.*

For planes that divide the complete pyramid into triangular pyramids will divide the frustum into triangular frustums having the altitude of the given frustum. If, now, a plane be passed so as to cut the given frustums in a section that is a mean proportional between the bases, it will cut each triangular frustum in a section that is a mean proportional between its upper and lower bases; that is, the mean proportional between the bases of the given frustum is the sum of those between the bases of the triangular frustums. The volume of the given frustum is the sum of the volumes of the triangular frustum; hence it is measured by the product of one third their common altitude into the sum of the upper and lower bases of the triangular frustums and the mean proportionals between them.

PROPOSITION XVII. THEOREM.

561. *Tetrahedrons with a trihedral angle of the one equal to a trihedral angle of the other, are to each other as the products of the edges of these trihedral angles.*



Given: V and V' , the volumes of two tetrahedrons having trihedral angle A of the one equal to trihedral angle A' of the other ;

To Prove: $V : V' = AB \times AC \times AS : A'B' \times A'C' \times A'S'$.

Apply one tetrahedron to the other so that $A' \neq A$.

From S and S' draw SP and $S'P'$, \perp to ABC and $A'B'C'$,
and let their plane intersect ABC in $AP'P$.

$$V : V' = ABC \times SP : A'B'C' \times S'P'. \quad (557)$$

$$\text{But } ABC : A'B'C' = AB \times AC : A'B' \times A'C', \quad (341)$$

$$\text{and } SP : S'P' = AS : A'S', \quad (289)$$

$$(\text{since } \triangle SAP \text{ is similar to } \triangle S'A'P'.) \quad (288)$$

Therefore, making the proper substitutions, we have,

$$V : V' = AB \times AC \times AS : A'B' \times A'C' \times A'S'. \quad \text{Q.E.D.}$$

562. *COR. Similar tetrahedrons are as the cubes of their homologous edges.*

For let $S-ABC$, $S'-A'B'C'$ be the similar tetrahedrons.

$$\text{Since } V : V' = SA \times SB \times SC : S'A' \times S'B' \times S'C', \quad (561)$$

$$\text{and } SA : S'A' = SB : S'B' = SC : S'C', \quad (\text{Hyp.})$$

$$V : V' = SA \times SA \times SA : S'A' \times S'A' \times S'A' = \overline{SA}^3 : \overline{S'A'}^3.$$

THE REGULAR POLYHEDRONS.

563. DEFINITION. A *regular polyhedron* has its faces all equal, regular polygons, and its polyhedral angles all equal.

PROPOSITION XVIII. THEOREM.

564. *There are only five possible regular polyhedrons.*

The faces of a regular polyhedron must be regular polygons (563); at least three are necessary to form each polyhedral angle (487); and the sum of the face angles of each polyhedral angle must be less than four right angles (492).

1°. The simplest regular polygon is the equilateral triangle, each of whose angles is 60° . Now

$$60^\circ \times 3 = 180^\circ, \quad 60^\circ \times 4 = 240^\circ, \quad \text{and} \quad 60^\circ \times 5 = 300^\circ;$$

but $60^\circ \times 6 = 360^\circ = 4 \text{ rt. } \angle$. Hence only three regular polyhedrons can have equilateral triangles as faces.

2°. The regular four-sided polygon, or square, has each of its angles 90° . Now

$$90^\circ \times 3 = 270^\circ, \quad \text{but} \quad 90^\circ \times 4 = 360^\circ = 4 \text{ rt. } \angle.$$

Hence only one regular polyhedron can be formed having squares as faces.

3°. The regular pentagon has each angle 108° . Now

$$108^\circ \times 3 = 324^\circ, \quad \text{but} \quad 108^\circ \times 4 = 432^\circ > 4 \text{ rt. } \angle.$$

Hence only one regular polyhedron can be formed having pentagons as faces.

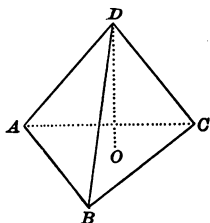
4°. The regular hexagon has each of its angles 120° . Now $120^\circ \times 3 = 360^\circ = 4 \text{ rt. } \angle$. Hence no regular polyhedron can be formed having as faces regular polygons of six or more sides. Hence only five regular polyhedrons can be formed: three having equilateral triangles as faces, — namely, the *tetrahedron*, the *octahedron*, and the *icosahedron*; one having squares as faces, — the *hexahedron* or *cube*; and one having pentagons as faces, — the *dodecahedron*.

PROPOSITION XIX. PROBLEM.

565. To construct the regular polyhedrons, an edge being given.

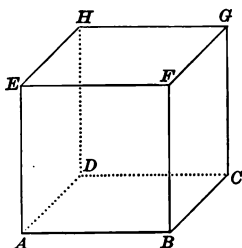
Given: A straight line AB as edge;

Required: To construct the regular polyhedrons.



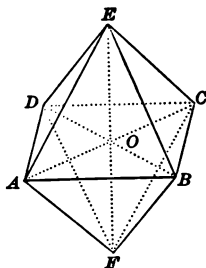
1°. Upon AB construct an equilateral triangle ABC , and find its center O . At O draw $OD \perp$ to ABC , and in OD take a point D such that $AD = AB$. Join DA, DB, DC ; $D-ABC$ is the required tetrahedron.

For the four faces are equilateral triangles (Const.); and the trihedral angles A, B, C, D , are equal (495), since their faces are all equal.

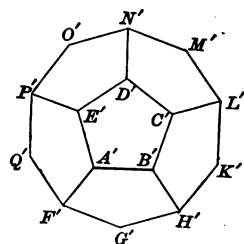
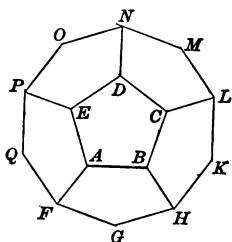


2°. Upon AB construct a square AC , and upon the sides of this square construct squares AF, BG, CH, DE , in planes perpendicular to the plane of AC ; AG is the required cube.

For the six faces are squares (Const.), and the trihedral angles A, B, C, D, E, F, G, H , are equal (495), since their faces are all equal.



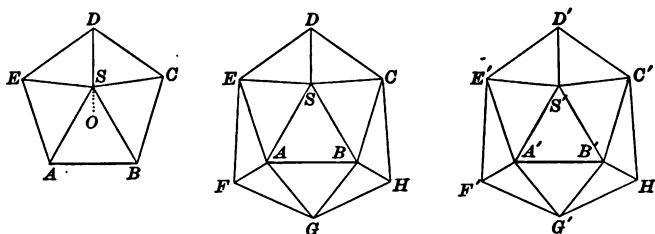
3°. Upon AB construct a square AC , and at its center O , draw $EF \perp$ to plane AC ; make $OF = OE = OA$, and join $EA, EB, EC, ED, FA, FB, FC, FD$. These edges are equal to each other and to OA , since $\triangle AOE, \triangle AOF$, etc., are equal right triangles; hence the faces of the figure are equal equilateral triangles. Also, since the triangles DEB, DFB , are equal, $DEBF$ is a square; whence it follows that the pyramid $A-DEBF$ is equal to the pyramid $E-ABCD$; hence the polyhedral angles A and E are equal; hence all the polyhedral angles of the figure are equal, and the figure is a regular octahedron.



4°. Upon AB construct a regular pentagon $ABCDE$, and to each side of $ABCDE$ apply an equal pentagon, so inclined to the plane of $ABCDE$ as to form trihedral angles at A, B ,

C, D, E . We thus obtain a convex surface $FHLNP$, composed of six regular pentagons. Construct a second convex surface $F'H'L'N'P'$, equal to the first, and apply it so as to form a single convex surface. This will contain the figure required.

For the faces are all equal pentagons (Const.), and the trihedral angles are equal, being contained by equal faces, and there being twelve such faces, the figure is a dodecahedron.



5°. Upon AB construct a regular pentagon $ABCDE$, and at its center O , draw $OS \perp$ to $ABCDE$. Take S so that $SA = AB$, and join SA, SB , etc., thus obtaining a regular pyramid $S-ABCDE$, having each of its faces an equilateral triangle. Now take the vertices A and B as the vertices of two other pyramids $A-BSEFG$ and $B-ASCHG$, having in common with $S-ABCDE$ the faces ASB, ASE , and ASB, BSC respectively, and in common with each other, the faces ASB and ABG . We thus obtain a convex surface $CDEFGH$, consisting of ten equal equilateral faces.

Construct a second convex surface $C'D'E'F'G'H'$, equal to the first, and apply it to the first so as to form a single convex surface. This will contain the figure required.

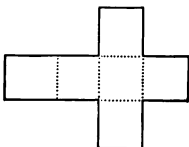
For any two consecutive face angles of the one will unite with any three consecutive face angles of the other so as to form a regular pentahedral angle. Hence the pentahedral angles are all equal, being formed by equal faces, and there are twenty such faces; hence they form an icosahedron.

SCHOLIUM. The regular polyhedrons may be constructed in the following way:

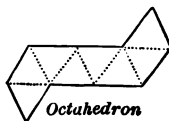
Having drawn on cardboard the diagrams below, cut through the heavy lines and half through the dotted lines;



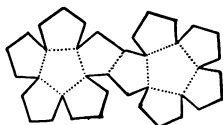
Tetrahedron



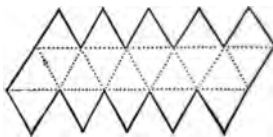
Hexahedron



Octahedron



Dodecahedron



Icosahedron

bring the edges together, and keep them in position by pasting over them strips of paper.



EXERCISES.

QUESTIONS.

743. What is the least number of points that can limit a line? The least number of straight lines that can limit a surface? The least number of plane surfaces that can limit a solid?

744. What is the lateral surface of a prism whose lateral edge is 25 in., and the perimeter of whose right section is 42 in.?

745. What is the volume of a right prism whose altitude is 40 in., and whose base contains $3\frac{1}{2}$ sq. ft.?

746. With what theorem of Book V. is Prop. IV. analogous?

747. With what theorems of Book I. are Prop. V. and its corollary analogous?

748. With what theorems of Book V. are Props. VI., VII., VIII., and IX. analogous?

749. How many square feet are there in the surface of a rectangular parallelopiped whose dimensions are 16 in., 22 in., and 30 in., respectively?

750. How many cubic feet in the volume of the same solid?

751. The dimensions of one rectangular parallelopiped are 2 ft., 5 ft., and 14 ft., respectively; those of another are 3 ft., 4 ft., and 10 ft., respectively. What is the ratio of the first solid to the second?

752. The volume of a rectangular parallelopiped is 96 cu. ft., and its altitude is 8 ft. 3 in. What is the area of its base?

753. The volume of a rectangular parallelopiped is 120 cu. ft., and the dimensions of its base are 4 ft. 5 in. and 6 ft. respectively. What is its altitude?

754. The edge of a cube is $4\frac{1}{2}$ in. What are its volume and its entire surface?

755. What should be the edge of a cubical box that shall contain a gallon dry measure?

756. What should be the edge of a cube so that its entire surface shall be a square foot?

757. What should be the altitude of a prism whose base is $3\frac{1}{2}$ sq. in., so that it may have the same volume as a prism whose altitude is $5\frac{1}{2}$ in. and whose base is $2\frac{3}{4}$ sq. in.?

758. The base of a pyramid is 12 sq. ft. and its altitude is 6 ft. What is the area of a section parallel to the base and 2 ft. from it?

759. The perimeter of the base of a pyramid is 15 in.; its slant height is 7 in. What is the lateral surface?

760. What is the volume of a pyramid whose base is 12.3 sq. in., and whose altitude is 5.72 ft.?

761. The bases of two pyramids are 6.4 sq. ft. and 8.1 sq. ft. respectively; their altitudes are 9 in. and 8 in. respectively. What is their ratio?

762. The bases of a frustum of a pyramid are 8 sq. ft. and $4\frac{1}{2}$ sq. ft. respectively, and its altitude is 5 ft. What is its volume?

763. The bases of a frustum of a pyramid are 20 sq. in. and 7.2 sq. in. respectively. Its volume is 400 cu. in. What is its altitude?

THEOREMS.

764. Every section of a prism by a plane parallel to the lateral edges is a parallelogram.

765. The lateral areas of right prisms having equal altitudes are as the perimeters of their bases.

766. The diagonals of a rectangular parallelepiped are equal.

767. The square of a diagonal of a rectangular parallelepiped is equal to the sum of the squares of the three diagonals meeting in any vertex.

768. The volume of a triangular prism is equal to one half the product of any lateral face by its distance from the opposite edge.

769. The volume of any prism is equal to the product of its right section by an edge.

770. The four diagonals of a parallelepiped bisect each other.

771. If the four diagonals of a four-sided prism pass through a common point, the prism is a parallelepiped.

772. Any straight line passing through the center of a parallelepiped and terminated by two faces, is bisected at the center.

N.B.—The center of a parallelepiped is the point of intersection of its diagonals.

773. Any plane passing through the center of a parallelepiped divides it into two equal solids.

774. If any two nonparallel diagonal planes of a prism are perpendicular to the base, the prism is a right prism.

775. The lateral surface of any pyramid is greater than its base.

776. The mid points of the edges of a regular tetrahedron are at the vertices of a regular octahedron.

777. The section of a triangular pyramid made by a plane passed parallel to two opposite edges is a parallelogram.

778. The section of a regular tetrahedron made by a plane passed parallel to two opposite edges is a rectangle.

779. The altitude of a regular tetrahedron is equal to the sum of the perpendiculars to the faces from any point within the figure.

BOOK IX.

THE THREE ROUND BODIES.



Of the solids that are bounded by curved surfaces, only three are treated of in Elementary Geometry, viz., the *cylinder*, the *cone*, and the *sphere*, usually referred to as the *three round bodies*.

CYLINDERS.

566. A *cylindrical surface* is a curved surface generated by a straight line that moves so as continually to touch a given curve, while remaining parallel to its original position. Thus if the straight line Aa moves so as to remain always parallel to its first position Aa , while continually touching the curve $ABCD$, the surface $ABCD-abcd$ thus generated is a cylindrical surface. The moving line is called the *generatrix*; the curve touched, the *directrix*; and any straight line Bb , that represents the generatrix in any of its positions, an *element* of the surface.

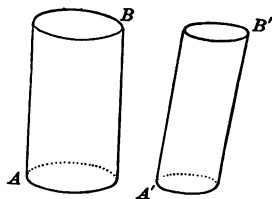


Since the generatrix is of indefinite length, a cylindrical surface may be regarded as extending indefinitely in two directions. As the directrix, moreover, may be a curve of any kind, close or not, the surface generated may present a corresponding variety of form. In elementary geometry, for obvious reasons, the directrix is usually assumed to be a circle.

567. A *cylinder* is a solid bounded by a cylindrical surface whose directrix is a closed curve, and by two parallel planes. These planes are called the *bases* of the cylinder; the curved surface, the *lateral surface*; and the perpendicular distance between the bases, the *altitude*.

From the definition it is evident that the curved surface of a cylinder must have as directrix a closed curve; since, otherwise, besides the two parallel bases, at least one other plane face would be needed in order to form a solid.

568. A *right cylinder* has its elements perpendicular to its base, as AB ; an *oblique cylinder* has its elements oblique to its base, as $A'B'$.



569. A *circular cylinder* is one that has a circle for each base. As only circular cylinders are treated of in this book, the term *cylinder* is to be understood as signifying *circular cylinder*.

570. A right cylinder with a circular base is called a *cylinder of revolution*, because it may be generated by the revolution of a rectangle about one of its sides as axis. This side is then called the *axis* of the cylinder, and the radius of the base, the *radius* of the cylinder.

EXERCISE 780. Only one straight line can be drawn through a given point on a cylindrical surface.

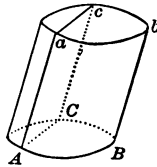
781. A straight line that joins two points on a cylindrical surface must coincide with an element of that surface.

782. If a plane contains one and only one straight line in common with the curved surface of a cylinder, the plane touches but does not intersect that surface.

DEFINITIONS. Such a plane is said to be *tangent* to the cylinder, and the common element is called the *element of contact*. A *tangent line* touches but does not intersect the cylinder.

PROPOSITION I. THEOREM.

571. *Every section of a cylinder made by a plane passing through an element is a parallelogram.*



Given: Ac , a section of cylinder $AB-c$ by a plane through element Aa ;

To Prove: Ac is a parallelogram.

For if through C a line be drawn \parallel to Aa ,
that line will be an element of the surface; (566)

it will also be a line in plane Ac ; (424)

\therefore the line will coincide with Cc , the intersection
of plane Ac and the lateral surface ;

$\therefore Cc$ is \parallel to Aa ,

and ac is \parallel to AC , (451)

(since they lie in parallel planes;)

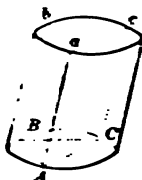
$\therefore Ac$ is a parallelogram. Q.E.D.

572. COR. *Every section of a right cylinder passing through an element is a rectangle.*

SCHOLIUM. It will be noticed that the properties established in Prop. I. and Prop. II., being independent of the form of the base, hold true, not only in regard to circular cylinders, but also to cylinders in general. A similar remark applies also to Prop. III. concerning the cone.

PROPOSITION II. THEOREM.

573. *The bases of a cylinder are equal.*



Given: ABC, abc , bases of cylinder Ac ;

To Prove: ABC equals abc .

Through any element Aa , pass planes forming the sections Ab, Ac ; and join BC, bc .

Since Ab and Ac are parallelograms, (571)

Bb and Cc are each \parallel and $=$ to Aa ; (136)

$\therefore Cb$ is a par'm. and $BC = bc$; (142)

$\therefore \triangle ABC = \triangle abc$. (69)

If, then, the upper base be applied to the lower, so that

$$ab \neq AB,$$

$$\text{then } \triangle abc \neq \triangle ABC,$$

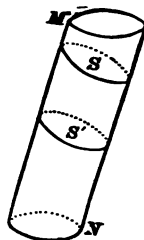
$$\text{and } c \neq C;$$

that is, any point c in the perimeter of the upper base will coincide with a corresponding point in the perimeter of the lower base;

\therefore the bases coincide and are equal. Q.E.D.

574. COR. 1. *Any two parallel sections, S, S' , cutting a cylindrical surface MN , are equal.*

575. COR. 2. *All sections of a circular cylinder parallel to the base are equal.*



CONES.

576. A *conical surface* is a curved surface generated by a straight line that moves so as continually to touch a given curve, and pass through a fixed point not in the plane of that curve. Thus if the straight line AA' moves so as always to touch the curve $ABCDE$, and pass through a fixed point S , the surface $S-ABCDE$ is a conical surface.

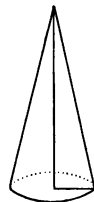
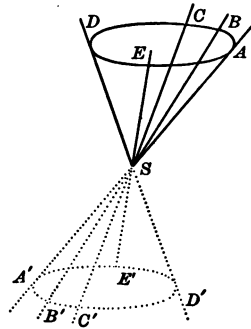
The moving line is called the *generatrix*; the curve it touches, the *directrix*; the fixed point, the *vertex*. Any straight line, as SB , representing one of the positions of the generatrix, is an *element* of the conical face.

If the generatrix is of indefinite length, the whole surface generated consists of two portions, each of indefinite extent, and lying on opposite sides of the vertex, the one being called the *upper nappe*, as $S-ABCDE$; the other, the *lower nappe*, as $S-A'B'C'D'E'$.

577. A *cone* is a solid bounded by a conical surface and a plane cutting that surface, as $S-ABCDE$. The plane surface is called the *base*; the curved surface, the *lateral surface*; S , the *vertex*; and the perpendicular distance from the vertex to the base, the *altitude*.

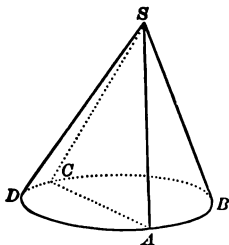
578. A *circular cone* is one that has a circular base. Its *axis* is the straight line drawn from the vertex to the center of the base.

579. A *right circular cone* has its axis perpendicular to the base; it is also called a *cone of revolution*, because it may be regarded as generated by the revolution of a right triangle about one of its arms as an axis.



PROPOSITION III. THEOREM.

580. *Every section of a cone made by a plane passing through the vertex is a triangle.*



Given : $S-AC$, a section of cone $S-ABD$, passing through vertex S ;

To Prove : $S-AC$ is a triangle.

The lines SA , SC , are elements of the cone ; (576)

they also lie in plane SAC , (9)

(since their extremities are in that plane ;)

$\therefore SA$, SC , are the intersections of the plane of section and the lateral surface.

AC is also a straight line, (426)

(being the intersection of two planes ;)

$\therefore SAC$ is a triangle.

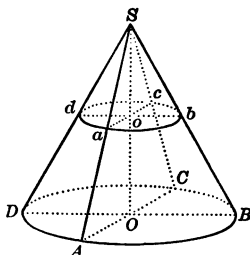
Q.E.D.

EXERCISE 783. A cylinder of revolution, with an altitude a and a radius r , rolls on a plane until the original element of contact again coincides with the plane. What is the figure of the plane surface passed over, and what is its area ?

784. A cone of revolution, altitude a , radius r , rolls on a plane, its vertex remaining fixed until the original element again coincides with the plane. Name the figure described, and give its area.

PROPOSITION IV. THEOREM.

581. *Every section of a circular cone made by a plane parallel to the base is a circle.*



Given: A section abd parallel to ABD , the base of circular cone $S-ABD$;

To Prove: abd is a circle.

Draw SO , the axis of the cone, cutting abd in o .

Through SO and any elements, SA , SB , etc., pass planes cutting the base in the radii OA , OB , etc., and abd in oa , ob , etc.

Since OA is \parallel to oa , and OB is \parallel to ob , etc., (451)

(being intersections of \parallel planes by a third plane,)

$\triangle Soa$, $\triangle Sob$, etc., are similar to $\triangle SOA$, $\triangle SOB$, etc., resp.; (292)

$\therefore oa : OA = so : SO = ob : OB$, etc.

But $OA = OB$; (162)

$\therefore oa = ob = oc$, etc.;

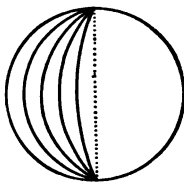
\therefore section abd is a circle, Q.E.D. (158)

(since all st. lines drawn from o to its perimeter are equal.)

582. COR. *The axis of a circular cone passes through the center of all sections that are parallel to the base.*

SPHERES.

583. A *sphere* is a solid bounded by a curved surface all the points of which are equally distant from a point within, called the *center*. Hence a sphere may be generated by the revolution of a semicircle about its diameter as axis.

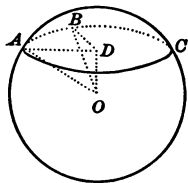


584. A *radius* of a sphere is any straight line drawn from the center to the surface. A *diameter* is any straight line drawn through the center and terminated both ways by the surface.

585. COR. All diameters of a sphere are equal.

PROPOSITION V. THEOREM.

586. Every section of a sphere made by a plane is a circle.



Given: A plane section ABC of a sphere whose center is O ;

To Prove: ABC is a circle.

- 1°. If the plane passes through the center O , then the lines drawn from O to any points, A, B , in the perimeter of the section, are equal, being radii of the sphere; therefore ABC is a circle. Q.E.D. (158)
- 2°. If the plane does not pass through O , draw $OD \perp$ to ABC , and join O and D with A and B , any points in the perimeter of the section.

Since radius $OA =$ radius OB , and OD is common,

rt. $\triangle ODA =$ rt. $\triangle ODB$ (72), and $DA = DB$;

$\therefore ABC$ is a circle, Q.E.D. (158)

(since any two points, A, B , in its perimeter, are equally distant from D .)

587. DEFINITION. A section that passes through the center is called a *great circle* of the sphere; a section not passing through the center, a *small circle* of the sphere.

588. DEFINITION. The diameter of a sphere that is perpendicular to the plane of a circle of the sphere is called the *axis* of that circle; the *poles* of the circle are the extremities of its axis.

589. COR. 1. *The axis of a circle of a sphere passes through the center of that circle; and conversely.*

For D , the foot of the perpendicular OD from the center of the sphere, is the center of the circle ABC .

590. COR. 2. *A circle nearer the center of a sphere is greater than one more remote.*

For the less the distance OD , the greater is DA , the radius of the circle.

591. COR. 3. *All great circles of a sphere are equal.*

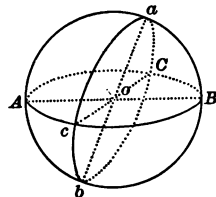
For their radii are radii of the sphere.

592. COR. 4. *Any two great circles, as ABC, abc , bisect each other.*

For since they have the same center o , their intersection Cc is a diameter of both and bisects both.

593. COR. 5. *Every great circle bisects the sphere.*

For the two parts into which the sphere is divided may



be placed so as to coincide, since otherwise there would be points on the surface unequally distant from the center.

594. COR. 6. *An arc of a great circle may be made to pass through any two given points A , C , on the surface of a sphere.*

For the two points, A and C , together with the center o , determine (424) the plane of a great circle whose circumference passes through A and C .

If the two points, as A and B , are the extremities of a diameter, since they are in the same straight line with the center o , any number of planes may be made to pass through them (421), and any number of semicircumferences.

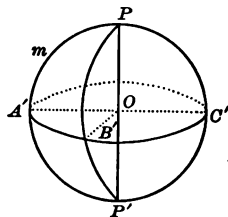
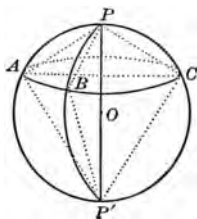
595. COR. 7. *One circle, and only one, can pass through three given points on the surface of a sphere.*

For these three points determine one plane (424), whose intersection with the sphere is a circle.

SCHOLIUM. Plane loci, as we have seen (211, etc.), are *lines* that may be regarded from two points of view, either as the path generated by *one* point moving in a certain way, or as the assemblage of *all* possible points having certain properties of position. *Loci in space*, again, are *surfaces* that may be regarded either as generated by *one* line moving in a certain way, or as the assemblage of *all* possible lines having certain properties of position. Thus (435) a plane may be regarded as generated by a line moving so as to remain always perpendicular to a given line at a given point. Cylindrical (566), conical (576), and spherical surfaces (583), have been defined as generated by the motion of certain lines moving in a certain way. From another point of view, the plane generated as above mentioned may be regarded as the locus of all perpendiculars to the given line at the given point; a circular cylindrical surface, as the locus of all parallels to a given line at a given distance; a circular conical surface, as the locus of all lines inclined to a given line, the axis, at a given angle.

PROPOSITION VI. THEOREM.

596. *All points in the circumference of a circle of a sphere are equally distant from each pole.*



Given: P, P' , poles of a circle ABC of the sphere whose center is O ;

To Prove: All points in the circumference ABC are equally distant from P or P' .

Join P with A, B, C , any points in circumf. ABC .

Since OP is \perp to circle ABC , (Hyp.)

OP passes through the center of ABC ; (589)

$\therefore PA = PB = PC$, (437)

(being obliques from P to points A, B, C , in plane ABC , that are equally distant from the \perp from P ;))

i.e., all points in circumference ABC are equally distant from P . Q.E.D.

For like reasons they are equally distant from P' .

597. COR. 1. *All arcs of great circles drawn from a pole of a circle to points in its circumference are equal.*

For their chords are equal chords (596) of equal circles (591).

598. DEFINITIONS. The *distance* between two given points on the surface of a sphere is the arc of a great circle joining those points; the *polar distance* of a point A in the cir-

cumference of a circle ABC is the arc of a great circle joining A and the nearer pole of ABC .

599. COR. 2. *The polar distance of a great circle $A'B'C'$ is a quadrant; that is, the fourth part of the circumference of a great circle.*

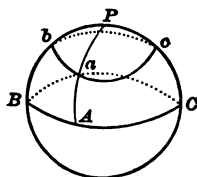
For the polar distance PmA' measures the rt. $\angle POA'$ (262).

600. COR. 3. *If a point P on the surface of a sphere is at the distance of a quadrant from any two points, A' , B' , in an arc of a great circle, then P is the pole of that circle.*

For the arcs PA' , PB' , being quadrants, the angles at O are right angles; therefore PO is \perp to OA' and to OB' ; hence it is \perp to the plane of arc $A'B'$ (428); whence P is the pole of arc $A'B'$ (588).

601. SCHOLIUM. A pole of a circle, great or small, whose circumference is to pass through a given point, being known, the circumference may be described on the surface of the sphere.

For by revolving Pa , an arc of a great circle, about the pole P , the extremity a will describe the small circle abc ; while the extremity of PA , a quadrant, will describe the great circle ABC . Or, placing one foot of the spherical compasses* at P , with an opening between the feet equal to the chord of the polar distance of the point a , we turn the second foot round the sphere so as to describe through a the required circumference.

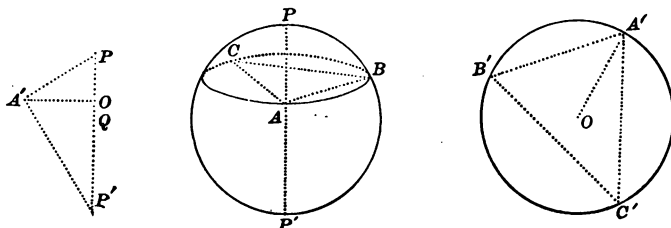


EXERCISE 785. Denoting by a and r the numerical measures of the axis and radius of the cylinder rolling as in Exercise 783, what must be the ratio of r to a so that the surface generated in one revolution shall be a square?

* Compasses with the feet inclined inwards.

PROPOSITION VII. PROBLEM.

602. To find the radius of a given sphere.



Given: A sphere APP' ;
Required: To find a radius of APP' .

Take any point P on the surface as pole, and with any opening of the compass describe a circumference ABC on the sphere.

Take any three points, A, B, C , in this circumference, and with the compasses take off the chord distances AB, AC, BC .

On any plane construct the triangle $A'B'C'$, having its sides equal to AB, AC, BC , respectively (205), and circumscribe a circumference about this triangle (185).

This circle will be equal to circle ABC , and its radius $A'O$ will be equal to the radius of circle ABC .

With $A'O$ as an arm, and $PA' =$ the chord of the arc joining PA , as hypotenuse, construct the rt. $\triangle PA'O$.

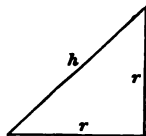
Draw $A'P' \perp$ to $A'P$, and produce it to meet PO in P' .

It is evident that PP' , thus determined, is equal to the diameter of the sphere, and its half, PQ , is the required radius.

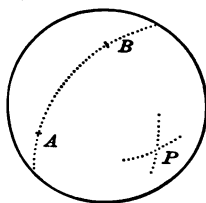
EXERCISE 786. If a cone of revolution roll upon a plane, its vertex remaining fixed, what kind of a surface is generated by the axis of the cone?

787. If a cone of revolution roll on the surface of a second cone, so that their vertices coincide, what kind of a surface is generated by the axis of the first cone?

603. SCHOLIUM 1. The radius of the sphere being known, we can obtain the chord of a quadrant of that sphere by finding the hypotenuse h of a right triangle having its arms each equal to r , the radius.



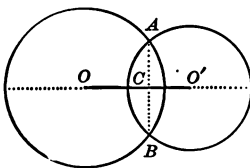
604. SCHOLIUM 2. The chord of a quadrant of a sphere being found, we can describe a circumference of a great circle through any two points, A, B , on the surface of the sphere, by describing from A and B as centers, quadrants intersecting in P , which will be the pole of the required circumference (600).



•

PROPOSITION VIII. THEOREM.

605. *The intersection of the surfaces of two spheres is the circumference of a circle perpendicular to the line of centers of the spheres.*



Given: OO' , the line of centers of two intersecting spheres ;

To Prove: The intersection of the surfaces is the circumference of a circle perpendicular to OO' .

Through the centers, O, O' , let a plane be passed, cutting the two spheres in great circles (587), which intersect in A and B .
(Hyp.)

Draw the chord AB , and produce OO' to meet the circumferences.

OO' is \perp to AB at its mid point C . (75)

If we now revolve the upper part of the figure about OO' , the two semicircles will generate the two spheres (583), while the point A will generate the line of intersection of the surfaces.

Also, since AC during the revolution remains \perp to OO' ,

AC will generate a circle whose center is C ;

i.e., the intersection of the surfaces of the spheres is the circumference of a circle \perp to OO' . Q.E.D.

606. DEFINITION. Two spheres are *tangent* if their surfaces have but one common point.

607. COR. If two spheres are tangent to each other, the point of tangency is in their line of centers.

For if we conceive the centers, O , O' , to remove farther from each other till the circumferences of the great circles become tangent, the points A and B will coincide with, and the circumference of intersection be reduced to, a point C .

608. SCHOLIUM. Two spheres being given in any relative position, a plane passed through their centers will cut them in great circles; and according as these circles are *within* or *without*, are *tangent* or *intersecting*, the spheres will have corresponding positions in regard to each other.

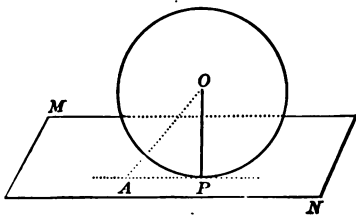
609. DEFINITION. A plane is *tangent* to a sphere when it has but one point in common with the surface of the sphere.

EXERCISE 788. Denoting by a and r the altitude and radius of a cone rolling as in Ex. 784, find the ratio of the base of the cone to the entire circle generated.

789. What fraction of that circle is described by one revolution of the cone ?

PROPOSITION IX. THEOREM.

610. *A plane perpendicular to a radius of a sphere at its extremity is tangent to the sphere.*



Given: OP , a radius of a sphere, and a plane $MN \perp$ to OP at its extremity;

To Prove: MN is tangent to the sphere.

Take any point except P in MN , as A , and join OA , PA .

Since OP is \perp to AP , (Hyp., 427)

$$OA > OP; \quad (99'')$$

$\therefore A$ lies without the sphere;

$\therefore MN$ is tangent to the sphere, Q.E.D. (609)

(since every point in MN , except P , lies without the sphere.)

611. COR. 1. *A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.*

612. COR. 2. *Any straight line drawn in the tangent plane through the point of contact is tangent to the sphere.*

613. COR. 3. *Any two straight lines tangent to the sphere at the point of contact determine the tangent plane at that point.*

SPHERICAL ANGLES AND POLYGONS.

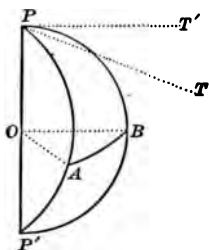
614. DEFINITION. The *angle* of two intersecting curves is the angle contained by the two tangents to the curves at the common point.

This definition applies, whatever be the surface upon which the curves are described.

615. DEFINITION. A *spherical angle* is the angle included between two arcs of great circles of a sphere. The arcs are its *sides*; their intersection is its *vertex*.

PROPOSITION X. THEOREM.

616. *A spherical angle is measured by the arc of a great circle described from its vertex as pole, and included by its sides, produced if necessary.*



Given: AB , an arc of a great circle described from the vertex of spherical angle APB as pole, and included between the sides of angle APB ;

To Prove: Spherical angle APB is measured by arc AB .

Draw PT , PT' , tangents to PAP' , PBP' respectively, and radii OA , OB .

Since PT is \perp to PP' in plane PAP' , (Const., 191)

and OA is \perp to PP' in plane PAP' ,

(PA being a quadrant,) (Hyp.)

PT is \parallel to OA ; }
similarly PT' is \parallel to OB ; } (106)

$\therefore \angle TPT' = \angle AOB$. (455)

But $\angle AOB$ is meas. by arc AB ; (262)

$\therefore \angle TPT'$ is meas. by arc AB ,

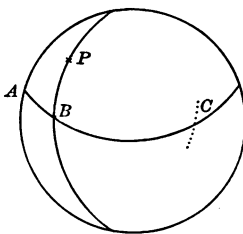
i.e., spher. $\angle APB$ is meas. by arc AB . Q.E.D. (614)

617. COR. 1. *A spherical angle has the same measure as the dihedral angle formed by the planes of its sides.*

618. COR. 2. *All arcs of great circles drawn through the pole of a given great circle are perpendicular to its circumference.*

For their planes are each perpendicular to its plane (469).

619. SCHOLIUM. The foregoing corollary enables us, through any given point P on the surface of a sphere, to describe an arc of a great circle perpendicular to a given arc ABC of a great circle. From P as pole describe (428) an arc of a great circle cutting ABC in C , and from C as pole describe an arc of a great circle passing through P and cutting ABC in B ; then arc PB is \perp to arc ABC (618).



620. DEFINITION. A *spherical polygon* is a portion of the surface of a sphere bounded by three or more arcs of great circles. The bounding arcs are the *sides* of the polygon; the angles formed by these sides are the *angles*, and the vertices of the angles are the *vertices*, of the polygon.

621. Since the planes of all great circles pass through the center of the sphere, the planes of the sides of a spherical polygon form at the center a polyhedral angle whose edges are radii drawn to the vertices of the polygon; the face angles of the polyhedral angle are angles at the center measured by the arcs that form the sides of the polygon; while the dihedral angles of the polyhedral angle have the same measure as the angles of the polygon (616).

Thus the planes of the sides of the polygon $ABCD$ (see diagram for Prop. XII.) form at O , the center of the sphere, the polyhedral angle $O-ABCD$. The face angles, AOB , BOC , etc., are measured by the sides AB , BC , etc., of the polygon; and the dihedral angle whose edge is the radius OA , has the same measure as the spherical angle BAD , etc.

622. From the relations thus established between polyhedral angles and spherical polygons, it clearly follows that *from any known property of polyhedral angles we may infer a corresponding property of spherical polygons; and conversely.*

623. DEFINITION. A *diagonal* of a spherical polygon is an arc of a great circle passing through two nonadjacent vertices.

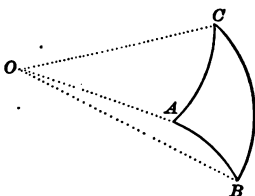
624. DEFINITION. A *spherical triangle* is a spherical polygon having three sides. Like plane triangles, spherical triangles may be *right* or *oblique*, *scalene*, *isosceles*, or *equilateral*.

EXERCISE 790. Out of a circle with radius R a sector of 60° is cut, and the edges of the remaining sector are joined so as to form the lateral surface of a cone of revolution. Find the radius r , and the altitude a , of the cone thus formed.

791. Find general formulas for the radius r and the altitude a of a cone of revolution whose lateral surface is formed from a sector that is $\frac{m}{n}$ ths of a circle with radius R .

PROPOSITION XI. THEOREM.

625. *Any side of a spherical triangle is less than the sum of the other two.*



Given : A spherical triangle ABC , on a sphere whose center is O ;

To Prove : $AB + AC$ is greater than BC .

In the trihedral angle $O-ABC$,

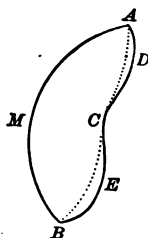
$$\angle AOB + \angle AOC > \angle BOC; \quad (491)$$

$$\therefore \text{arc } AB + \text{arc } AC > \text{arc } BC. \quad \text{Q.E.D.} \quad (616)$$

626. COR. 1. *Any side of a spherical triangle is greater than the difference of the other two.*

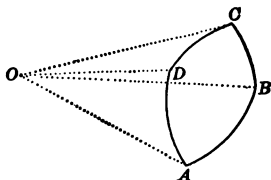
627. COR. 2. *The shortest path on the surface of a sphere between two given points, A and B , is the arc AMB of a great circle passing through those points.*

For let ACB be any other curve joining A and B . Take in it any point C , and through A , C , and C , B , describe arcs of great circles (594). Then $AMB < AC + CB$ (625). Between A and C , C and B , take any points, D and E , and describe arcs of great circles joining AD , DC , CE , EB ; then $AC + CB < AD + DC + CE + EB$. If this process be continued indefinitely, the sum of the arcs thus obtained will be always increasing, and approaching the curve ACB as its limit; hence ACB must be greater than AMB .



PROPOSITION XII. THEOREM.

628. *The sum of the sides of any spherical polygon is less than the circumference of a great circle.*



Given: A spherical polygon $ABCD$, on a sphere with center O ;

To Prove: $AB + BC + CD + DA < \text{a circumf. of a great circle.}$

Draw the radii OA, OB, OC, OD .

In the polyhedral angle $O-ABCD$,

$\angle AOB + \angle BOC + \angle COD + \angle DOA < \text{four rt. } \angle\text{'s};$ (492)

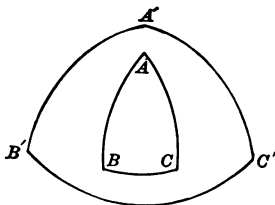
$\therefore \text{arc } AB + \text{arc } BC + \text{arc } CD + \text{arc } DA < \text{a circumf. of a great } \odot.$

Q.E.D. (616)



POLAR TRIANGLES.

629. DEFINITION. If from the vertices of a spherical triangle as poles, arcs of great circles be described, they will form by their intersection a second triangle, called the *polar triangle* of the first. Thus if A, B, C , the vertices of the spherical triangle ABC , are the poles of the arcs $B'C', A'C', A'B'$, forming the spherical triangle $A'B'C'$, then $A'B'C'$ is the polar triangle of ABC .



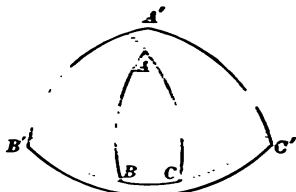
Since all circumferences of great circles intersect in two

joining the arcs AB , AC , $B'C'$ if produced will form three* other triangles on the surface of the sphere. Hence it is to be understood that the triangle to be taken as the polar triangle of ABC is the central one whose vertex A' , homologous to A is on the same side of BC as the vertex A , and similarly in regard to the other vertices.

—◆—

PROPOSITION XIII. THEOREM.

630. *If the first of two spherical triangles is the polar triangle of the second, reciprocally, the second triangle is the polar triangle of the first.*



Given: $A'B'C'$, the polar triangle of ABC ;

To Prove: ABC is the polar triangle of $A'B'C'$.

Since B is a pole of $A'C'$, an arc of a great \odot , (Hyp.)
the distance of B from A' is a quadrant. (599)

Since C is a pole of $A'B'$, an arc of a great \odot ,
the distance of C from A' is a quadrant;

$\therefore A'$ is a pole of BC , (600)

(being at a quadrant's distance from its extremities.)

Similarly, B' and C' are poles of AC and AB resp.

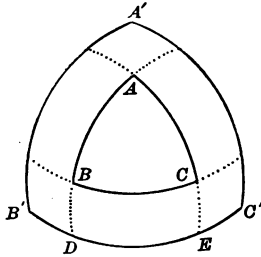
Also, A' and A are on the same side of BC , etc.;

$\therefore ABC$ is the polar triangle of $A'B'C'$. Q.E.D. (629)

* Excluding those having sides greater than a semicircumference.

PROPOSITION XIV. THEOREM.

631. In two polar triangles, each angle of the one is measured by the supplement of the opposite side of the other.



Given: ABC and $A'B'C'$, two polar triangles;

To Prove: Angle A is measured by $180^\circ - \text{arc } B'C'$, etc.

Produce AB , AC , if necessary, to meet $B'C'$ in D , E resp.

Since A is the pole of arc DE , (Hyp.)

$\angle A$ is measured by arc DE . (616)

Since B' and C' are poles of arcs AE , AD , resp., (Hyp.)

arcs $B'E$ and $C'D$ are each quadrants;

$\therefore B'E + C'D = \text{a semicircumf.};$

$\therefore B'C' + DE = \text{a semicircumf.}, = 180^\circ;$

$\therefore DE$, which measures $\angle A$, $= 180^\circ - B'C'$.

In the same way it may be proved that $\angle B$ and C are measured by the supplements of arcs $A'C'$ and $A'B'$ resp. Q.E.D.

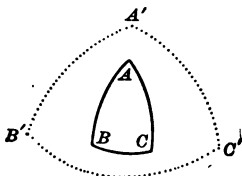
632. DEFINITION. Polar triangles are also called *supplemental triangles*. For if we denote by a' the number of degrees in $B'C'$, the side of $A'B'C'$ that is opposite $\angle A$, etc., we have the relations:

$$\angle A = 180^\circ - a', \quad \angle B = 180^\circ - b', \quad \angle C = 180^\circ - c',$$

$$\angle A' = 180^\circ - a, \quad \angle B' = 180^\circ - b, \quad \angle C' = 180^\circ - c.$$

PROPOSITION XV. THEOREM.

633. *The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.*



Given: A, B, C , the angles of a spherical triangle ABC ;

To Prove: $\angle A + \angle B + \angle C > 180^\circ$ and $< 540^\circ$.

Construct $A'B'C'$, the polar triangle of ABC . Then denoting (as in Art. 632) the number of degrees in $B'C'$, $A'C'$, $A'B'$, resp. by a' , b' , c' ,

since $\angle A = 180^\circ - a'$, $\angle B = 180^\circ - b'$, $\angle C = 180^\circ - c'$, (632)

$$\angle A + \angle B + \angle C = 540^\circ - (a' + b' + c'). \quad (\text{Ax. 2})$$

$$\text{But } a' + b' + c' < 360^\circ \text{ and } > 0^\circ; \quad (492)$$

$$\therefore \angle A + \angle B + \angle C > (540^\circ - 360^\circ) \text{ or } 180^\circ,$$

$$\text{and } \angle A + \angle B + \angle C < (540^\circ - 0^\circ) \text{ or } 540^\circ. \quad \text{Q.E.D.}$$

634. COR. *A spherical triangle may have two, or even three, right angles; also two, or even three, obtuse angles.*

635. DEFINITION. A spherical triangle having two right angles is said to be *birectangular*; a spherical triangle having three right angles is said to be *trirectangular*.

636. DEFINITION. The excess of the sum of the angles of a spherical triangle over two right angles is called the *spherical excess* of the triangle.

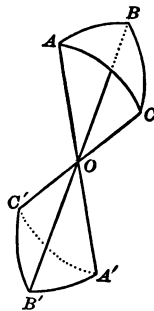
Thus, denoting its spherical excess by E , we have for $\triangle ABC$,

$$E = \angle A + \angle B + \angle C - 180^\circ.$$

637. DEFINITION. The *spherical excess* of any spherical polygon of n sides is equal to the excess of the sum of its angles over $2(n - 2)$ right angles; that is, is equal to the sum of the spherical excess of the $(n - 2)$ spherical triangles into which the polygon can be divided by means of arcs drawn from any vertex to the opposite vertices.

638. DEFINITION. *Symmetrical spherical triangles* are such as have their sides severally equal, but arranged in reverse order; as ABC , $A'B'C'$.

$A'B'C'$ may be regarded as formed by producing AO , BO , CO , to meet the surface of the sphere in A' , B' , C' , and joining the points thus obtained by arcs of great circles. We can conceive $A'B'C'$, when thus formed, as moved to any other position on the spherical surface; and *vice versa*.



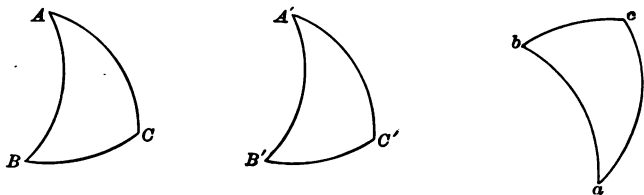
Since the face angles of the trihedrals $O-ABC$, $O-A'B'C'$, are equal, but arranged in reverse order, the trihedrals are symmetrical (493), and cannot be made to coincide unless isosceles (497). Hence the spherical triangles whose sides are the intersections of the planes of the faces of the trihedrals, cannot be made to coincide unless these trihedrals can be made to coincide; that is, unless they are isosceles.

639. COR. *An isosceles spherical triangle can be made to coincide with its symmetrical triangle.*

For as the trihedral angles formed by the planes of their sides can be made to coincide (497), the sides formed by the intersections of those planes with the surface can be made to coincide.

PROPOSITION XVI. THEOREM.

640. *Two triangles on the same sphere are either equal or symmetrical, if a side and the including angles of the one are respectively equal to a side and the including angles of the other.*



Given: In spherical triangles ABC , $A'B'C'$, AB equal to $A'B'$, angle A equal to angle A' , and angle B equal to angle B' ;

To Prove: ABC and $A'B'C'$ are either equal or symmetrical.

For $\triangle ABC$ may be placed either upon $\triangle A'B'C'$ or upon $\triangle abc$, symmetrical with $A'B'C'$, so as to coincide, as may be shown by the same course of reasoning as that employed in Prop. VI., Book I. Hence $\triangle ABC$ is either equal to $\triangle A'B'C'$ or symmetrical with it. Q.E.D.

641. COR. *If a spherical triangle has two equal angles, it is isosceles.*

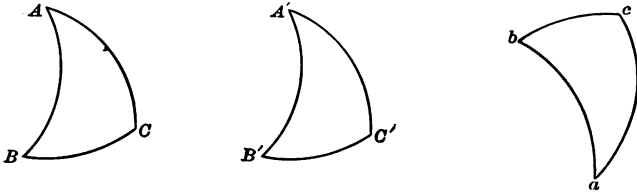
For if $\angle A = \angle B = \angle A' = \angle B'$, then, also, since $\angle A = \angle a$ and $\angle B = \angle b$, $\triangle ABC$ can be made to coincide with both $\triangle A'B'C'$ and with its symmetrical triangle abc . Hence $A'B'C'$ and abc must be isosceles (639); whence also ABC is isosceles.

EXERCISE 792. If two sides of a spherical triangle are quadrants, the third side measures the opposite angle.

793. A spherical triangle ABC has $\angle A = 83^\circ$, $\angle B = 50^\circ$, and $\angle C = 97^\circ$; find how many degrees there are in the sides of the polar triangle that are respectively opposite to those angles.

PROPOSITION XVII. THEOREM.

642. *Two triangles on the same sphere are either equal or symmetrical, if two sides and the included angle of the one are respectively equal to two sides and the included angle of the other.*



Given: In spherical triangles ABC , $A'B'C'$, AB equal to $A'B'$, AC equal to $A'C'$, and angle A equal to angle A' ;

To Prove: ABC and $A'B'C'$ are either equal or symmetrical.

For $\triangle ABC$ can be made to coincide either with $\triangle A'B'C'$, or with its symmetrical triangle abc , as may be shown by the same course of reasoning as that employed in Prop. VIII., Book I. Hence $\triangle ABC$ is either equal to $\triangle A'B'C'$ or symmetrical with it. Q.E.D.

643. COR. 1. *In an isosceles spherical triangle, the angles opposite the equal sides are equal.*

For if ABC is isosceles, then both $A'B'C'$ and its symmetrical triangle abc must be isosceles. Hence, as $\angle B$ can be made to coincide with $\angle c$, and $\angle c = \angle C' = \angle C$, $\angle B = \angle C$.

644. COR. 2. *The arc of a great circle drawn from the vertex of an isosceles spherical triangle to the mid point of the base, bisects the vertical angle, is perpendicular to the base, and divides the triangle into two symmetrical triangles.*

EXERCISE 794. If a spherical triangle ABC is isosceles, its polar triangle $A'B'C'$ is also isosceles.

PROPOSITION XVIII. THEOREM.

645. *Two triangles on the same sphere are either equal or symmetrical, if the three sides of the one are respectively equal to the three sides of the other.*

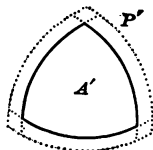
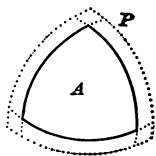
For as the respectively equal sides must be arranged either in the same or in the reverse order, the one triangle can be made to coincide either with the other or with its symmetrical triangle, as may be shown by a course of reasoning similar to that employed in Prop. X., Book I. Q.E.D.

646. SCHOLIUM. This and other theorems in regard to triangles, etc., on the same sphere, will evidently hold true in regard to triangles, etc., on equal spheres; since, if the centers of the equal spheres be made to coincide, their surfaces will also coincide.



PROPOSITION XIX. THEOREM.

647. *If two triangles on the same sphere are mutually equiangular, they are mutually equilateral and are either equal or symmetrical.*



Given: Two mutually equiangular spherical triangles A and A' ;

To Prove: A and A' are mutually equilateral and either equal or symmetrical.

Construct P, P' , the polar triangles of A, A' respectively.

Since A and A' are mutually equiangular, (Hyp.)

P and P' , their polar Δ , are mutually equilateral, (631)
(their corresponding angles being measures of equal angles;)

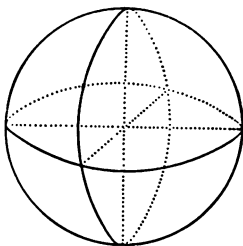
$\therefore P$ and P' are mutually equiangular; (645)

$\therefore A$ and A' , the polar Δ of P and P' , are mutually equilateral; (631)

$\therefore A$ and A' are equal or symmetrical. Q.E.D. (645)

SCHOLIUM. Mutually equiangular spherical triangles are equilateral only when on the same sphere or equal spheres. When the equiangular triangles are on unequal spheres, their homologous sides are proportional to the radii of their spheres; the triangles are then said to be *similar*.

648. COR. If three planes are passed through the center of a sphere, each perpendicular to the other two, they divide the surface into eight equal trirectangular triangles (621, 647).



EXERCISE 795. Birectangular triangles on equal spheres are equal if their acute angles are equal.

796. Birectangular triangles on unequal spheres are similar if their acute angles are equal.

797. Trirectangular triangles on equal spheres are equal.

798. Trirectangular triangles on unequal spheres are similar.

799. The polar triangle of a birectangular triangle is birectangular.

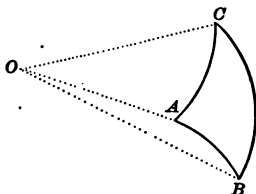
800. In a birectangular triangle, the sides that are opposite to the right angles are quadrants.

801. Each side of a trirectangular triangle is a quadrant.

802. The polar triangle of a trirectangular triangle is a trirectangular triangle coinciding with the triangle itself.

PROPOSITION XI. THEOREM.

625. *Any side of a spherical triangle is less than the sum of the other two.*



Given : A spherical triangle ABC , on a sphere whose center is O ;

To Prove : $AB + AC$ is greater than BC .

In the trihedral angle $O-ABC$,

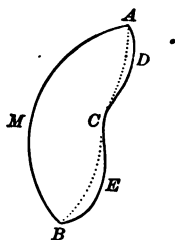
$$\angle AOB + \angle AOC > \angle BOC ; \quad (491)$$

$$\therefore \text{arc } AB + \text{arc } AC > \text{arc } BC. \quad \text{Q.E.D.} \quad (616)$$

626. COR. 1. *Any side of a spherical triangle is greater than the difference of the other two.*

627. COR. 2. *The shortest path on the surface of a sphere between two given points, A and B , is the arc AMB of a great circle passing through those points.*

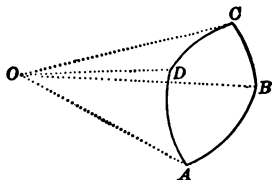
For let ACB be any other curve joining A and B . Take in it any point C , and through A , C , and C , B , describe arcs of great circles (594). Then $AMB < AC + CB$ (625). Between A and C , C and B , take any points, D and E , and describe arcs of great circles joining AD , DC , CE , EB ; then $AC + CB < AD + DC + CE + EB$. If this



process be continued indefinitely, the sum of the arcs thus obtained will be always increasing, and approaching the curve ACB as its limit; hence ACB must be greater than AMB .

PROPOSITION XII. THEOREM.

628. *The sum of the sides of any spherical polygon is less than the circumference of a great circle.*



Given: A spherical polygon $ABCD$, on a sphere with center O ;

To Prove: $AB + BC + CD + DA < \text{a circumf. of a great circle.}$

Draw the radii OA, OB, OC, OD .

In the polyhedral angle $O-ABCD$,

$\angle AOB + \angle BOC + \angle COD + \angle DOA < \text{four rt. } \angle$; (492)

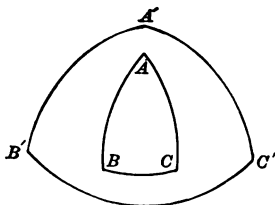
$\therefore \text{arc } AB + \text{arc } BC + \text{arc } CD + \text{arc } DA < \text{a circumf. of a great } \odot$.

Q.E.D. (616)



POLAR TRIANGLES.

629. DEFINITION. If from the vertices of a spherical triangle as poles, arcs of great circles be described, they will form by their intersection a second triangle, called the *polar triangle* of the first. Thus if A, B, C , the vertices of the spherical triangle ABC , are the poles of the arcs $B'C', A'C', A'B'$, forming the spherical triangle $A'B'C'$, then $A'B'C'$ is the polar triangle of ABC .



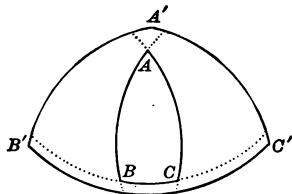
Since all circumferences of great circles intersect in two

points, the arcs $A'B'$, $A'C'$, $B'C'$, if produced, will form three* other triangles on the surface of the sphere. Hence it is to be understood that the triangle to be taken as the polar triangle of ABC is the central one, whose vertex A' , homologous to A , is on the same side of BC as the vertex A , and similarly in regard to the other vertices.



PROPOSITION XIII. THEOREM.

630. *If the first of two spherical triangles is the polar triangle of the second, reciprocally, the second triangle is the polar triangle of the first.*



Given: $A'B'C'$, the polar triangle of ABC ;

To Prove: ABC is the polar triangle of $A'B'C'$.

Since B is a pole of $A'C'$, an arc of a great \odot , (Hyp.)
the distance of B from A' is a quadrant. (599)

Since C is a pole of $A'B'$, an arc of a great \odot ,
the distance of C from A' is a quadrant;

$\therefore A'$ is a pole of BC , (600)

(being at a quadrant's distance from its extremities.)

Similarly, B' and C' are poles of AC and AB resp.

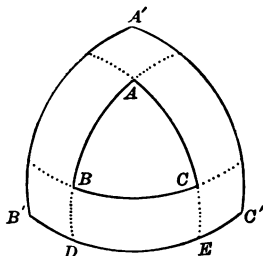
Also, A' and A are on the same side of BC , etc.;

$\therefore ABC$ is the polar triangle of $A'B'C'$. Q.E.D. (629)

* Excluding those having sides greater than a semicircumference.

PROPOSITION XIV. THEOREM.

631. In two polar triangles, each angle of the one is measured by the supplement of the opposite side of the other.



Given: ABC and $A'B'C'$, two polar triangles;

To Prove: Angle A is measured by $180^\circ - \text{arc } B'C'$, etc.

Produce AB , AC , if necessary, to meet $B'C'$ in D , E resp.

Since A is the pole of arc DE , (Hyp.)

$\angle A$ is measured by arc DE . (616)

Since B' and C' are poles of arcs AE , AD , resp., (Hyp.)

arcs $B'E$ and $C'D$ are each quadrants;

$\therefore B'E + C'D = \text{a semicircumf.};$

$\therefore B'C' + DE = \text{a semicircumf.}, = 180^\circ;$

$\therefore DE$, which measures $\angle A$, $= 180^\circ - B'C'$.

In the same way it may be proved that $\angle B$ and C are measured by the supplements of arcs $A'C'$ and $A'B'$ resp. Q.E.D.

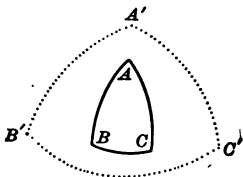
632. DEFINITION. Polar triangles are also called *supplemental triangles*. For if we denote by a' the number of degrees in $B'C'$, the side of $A'B'C'$ that is opposite $\angle A$, etc., we have the relations:

$$\angle A = 180^\circ - a', \quad \angle B = 180^\circ - b', \quad \angle C = 180^\circ - c',$$

$$\angle A' = 180^\circ - a, \quad \angle B' = 180^\circ - b, \quad \angle C' = 180^\circ - c.$$

PROPOSITION XV. THEOREM.

633. *The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.*



Given: A, B, C , the angles of a spherical triangle ABC ;

To Prove: $\angle A + \angle B + \angle C > 180^\circ$ and $< 540^\circ$.

Construct $A'B'C'$, the polar triangle of ABC . Then denoting (as in Art. 632) the number of degrees in $B'C'$, $A'C'$, $A'B'$, resp. by a' , b' , c' ,

since $\angle A = 180^\circ - a'$, $\angle B = 180^\circ - b'$, $\angle C = 180^\circ - c'$, (632)

$$\angle A + \angle B + \angle C = 540^\circ - (a' + b' + c'). \quad (\text{Ax. 2})$$

$$\text{But } a' + b' + c' < 360^\circ \text{ and } > 0^\circ; \quad (492)$$

$$\therefore \angle A + \angle B + \angle C > (540^\circ - 360^\circ) \text{ or } 180^\circ,$$

$$\text{and } \angle A + \angle B + \angle C < (540^\circ - 0^\circ) \text{ or } 540^\circ. \quad \text{Q.E.D.}$$

634. COR. *A spherical triangle may have two, or even three, right angles; also two, or even three, obtuse angles.*

635. DEFINITION. A spherical triangle having two right angles is said to be *birectangular*; a spherical triangle having three right angles is said to be *trirectangular*.

636. DEFINITION. The excess of the sum of the angles of a spherical triangle over two right angles is called the *spherical excess* of the triangle.

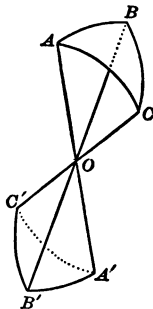
Thus, denoting its spherical excess by E , we have for $\triangle ABC$,

$$E = \angle A + \angle B + \angle C - 180^\circ.$$

637. DEFINITION. The *spherical excess* of any spherical polygon of n sides is equal to the excess of the sum of its angles over $2(n - 2)$ right angles; that is, is equal to the sum of the spherical excess of the $(n - 2)$ spherical triangles into which the polygon can be divided by means of arcs drawn from any vertex to the opposite vertices.

638. DEFINITION. *Symmetrical spherical triangles* are such as have their sides severally equal, but arranged in reverse order; as ABC , $A'B'C'$.

$A'B'C'$ may be regarded as formed by producing AO , BO , CO , to meet the surface of the sphere in A' , B' , C' , and joining the points thus obtained by arcs of great circles. We can conceive $A'B'C'$, when thus formed, as moved to any other position on the spherical surface; and *vice versa*.



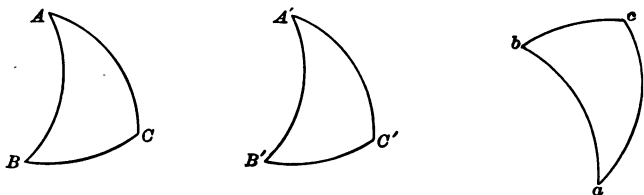
Since the face angles of the trihedrals $O-ABC$, $O-A'B'C'$, are equal, but arranged in reverse order, the trihedrals are symmetrical (493), and cannot be made to coincide unless isosceles (497). Hence the spherical triangles whose sides are the intersections of the planes of the faces of the trihedrals, cannot be made to coincide unless these trihedrals can be made to coincide; that is, unless they are isosceles.

639. COR. *An isosceles spherical triangle can be made to coincide with its symmetrical triangle.*

For as the trihedral angles formed by the planes of their sides can be made to coincide (497), the sides formed by the intersections of those planes with the surface can be made to coincide,

PROPOSITION XVI. THEOREM.

640. *Two triangles on the same sphere are either equal or symmetrical, if a side and the including angles of the one are respectively equal to a side and the including angles of the other.*



Given: In spherical triangles ABC , $A'B'C'$, AB equal to $A'B'$, angle A equal to angle A' , and angle B equal to angle B' ;

To Prove: ABC and $A'B'C'$ are either equal or symmetrical.

For $\triangle ABC$ may be placed either upon $\triangle A'B'C'$ or upon $\triangle abc$, symmetrical with $A'B'C'$, so as to coincide, as may be shown by the same course of reasoning as that employed in Prop. VI., Book I. Hence $\triangle ABC$ is either equal to $\triangle A'B'C'$ or symmetrical with it. Q.E.D.

641. COR. *If a spherical triangle has two equal angles, it is isosceles.*

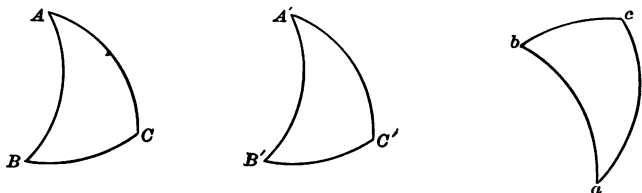
For if $\angle A = \angle B = \angle A' = \angle B'$, then, also, since $\angle A = \angle a$ and $\angle B = \angle b$, $\triangle ABC$ can be made to coincide with both $\triangle A'B'C'$ and with its symmetrical triangle abc . Hence $A'B'C'$ and abc must be isosceles (639); whence also ABC is isosceles.

EXERCISE 792. If two sides of a spherical triangle are quadrants, the third side measures the opposite angle.

793. A spherical triangle ABC has $\angle A = 83^\circ$, $\angle B = 50^\circ$, and $\angle C = 97^\circ$; find how many degrees there are in the sides of the polar triangle that are respectively opposite to those angles.

PROPOSITION XVII. THEOREM.

642. *Two triangles on the same sphere are either equal or symmetrical, if two sides and the included angle of the one are respectively equal to two sides and the included angle of the other.*



Given: In spherical triangles ABC , $A'B'C'$, AB equal to $A'B'$, AC equal to $A'C'$, and angle A equal to angle A' ;

To Prove: ABC and $A'B'C'$ are either equal or symmetrical.

For $\triangle ABC$ can be made to coincide either with $\triangle A'B'C'$, or with its symmetrical triangle abc , as may be shown by the same course of reasoning as that employed in Prop. VIII., Book I. Hence $\triangle ABC$ is either equal to $\triangle A'B'C'$ or symmetrical with it. Q.E.D.

643. COR. 1. *In an isosceles spherical triangle, the angles opposite the equal sides are equal.*

For if ABC is isosceles, then both $A'B'C'$ and its symmetrical triangle abc must be isosceles. Hence, as $\angle B$ can be made to coincide with $\angle c$, and $\angle c = \angle C' = \angle C$, $\angle B = \angle C$.

644. COR. 2. *The arc of a great circle drawn from the vertex of an isosceles spherical triangle to the mid point of the base, bisects the vertical angle, is perpendicular to the base, and divides the triangle into two symmetrical triangles.*

EXERCISE 794. If a spherical triangle ABC is isosceles, its polar triangle $A'B'C'$ is also isosceles.

PROPOSITION XVIII. THEOREM.

645. *Two triangles on the same sphere are either equal or symmetrical, if the three sides of the one are respectively equal to the three sides of the other.*

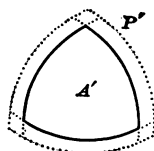
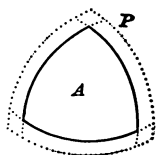
For as the respectively equal sides must be arranged either in the same or in the reverse order, the one triangle can be made to coincide either with the other or with its symmetrical triangle, as may be shown by a course of reasoning similar to that employed in Prop. X., Book I. Q.E.D.

646. SCHOLIUM. This and other theorems in regard to triangles, etc., on the same sphere, will evidently hold true in regard to triangles, etc., on equal spheres; since, if the centers of the equal spheres be made to coincide, their surfaces will also coincide.



PROPOSITION XIX. THEOREM.

647. *If two triangles on the same sphere are mutually equiangular, they are mutually equilateral and are either equal or symmetrical.*



Given: Two mutually equiangular spherical triangles A and A' ;

To Prove: A and A' are mutually equilateral and either equal or symmetrical.

Construct P, P' , the polar triangles of A, A' respectively.

Since A and A' are mutually equiangular, (Hyp.)

P and P' , their polar Δ , are mutually equilateral, (631)
(their corresponding angles being measures of equal angles;)

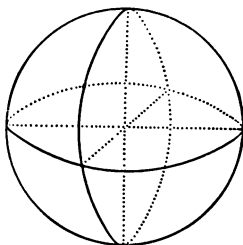
$\therefore P$ and P' are mutually equiangular; (645)

$\therefore A$ and A' , the polar Δ of P and P' , are mutually equilateral; (631)

$\therefore A$ and A' are equal or symmetrical. Q.E.D. (645)

SCHOLIUM. Mutually equiangular spherical triangles are equilateral only when on the same sphere or equal spheres. When the equiangular triangles are on unequal spheres, their homologous sides are proportional to the radii of their spheres; the triangles are then said to be *similar*.

648. COR. *If three planes are passed through the center of a sphere, each perpendicular to the other two, they divide the surface into eight equal trirectangular triangles (621, 647).*



EXERCISE 795. Birectangular triangles on equal spheres are equal if their acute angles are equal.

796. Birectangular triangles on unequal spheres are similar if their acute angles are equal.

797. Trirectangular triangles on equal spheres are equal.

798. Trirectangular triangles on unequal spheres are similar.

799. The polar triangle of a birectangular triangle is birectangular.

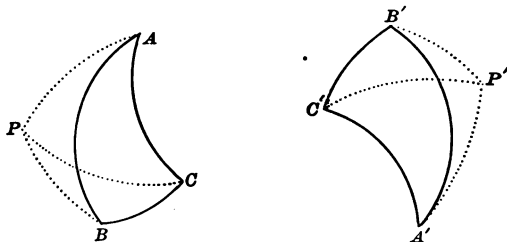
800. In a birectangular triangle, the sides that are opposite to the right angles are quadrants.

801. Each side of a trirectangular triangle is a quadrant.

802. The polar triangle of a trirectangular triangle is a trirectangular triangle coinciding with the triangle itself.

PROPOSITION XX. THEOREM.

649. *Symmetrical spherical triangles are equivalent.*



Given: In symmetrical triangles ABC , $A'B'C'$, AB equal to $A'B'$, AC equal to $A'C'$, BC equal to $B'C'$;

To Prove: Triangle ABC is equivalent to triangle $A'B'C'$.

Find P and P' , the poles of small circles passing through A, B, C , and A', B', C' respectively.

Since arcs $AB, AC, BC =$ arcs $A'B', A'C', B'C'$ resp., (Hyp.) the chords of arcs $AB, AC, BC =$ the chords of arcs $A'B', A'C', B'C'$ resp.;

\therefore the plane triangles formed by these chords are equal; (69)

\therefore the small circles through A, B, C , and A', B', C' , are equal. Through $PA, PB, PC, P'A', P'B', P'C'$, pass arcs of great circles.

Since arcs $PA, PB, PC, P'A', P'B', P'C'$, are equal, (596) (being polar distances of equal circles on the same sphere,)

$\triangle PAB, P'A'B'$, are symmetrical and isosceles;

$$\therefore \triangle PAB = \triangle P'A'B'. \quad (639)$$

Similarly, $\triangle PBC = \triangle P'B'C'$, and $\triangle PAC = \triangle P'A'C'$;

$$\therefore PAC + PBC - PAB = P'A'C' + P'B'C' - P'A'B';$$

(Ax. 2, Ax. 3.)

$$\therefore \triangle ABC = \triangle A'B'C'.$$

Q.E.D.

If pole P should lie within $\triangle ABC$, then pole P' would also lie within $\triangle A'B'C'$, and we should have

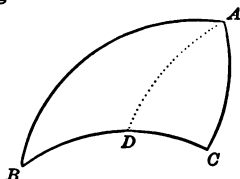
$$PAB + PAC + PBC \approx P'A'B' + P'A'C' + P'B'C';$$

$$\therefore \triangle ABC \approx \triangle A'B'C'.$$



PROPOSITION XXI. THEOREM.

650. In a spherical triangle, a greater side is opposite a greater angle.



Given: In spherical triangle ABC , angle A greater than angle B ;

To Prove: BC is greater than AC .

Through A describe AD , an arc of a great \odot , so that $\angle BAD = \angle B$.

$$\text{Since } \angle BAD = \angle B, \quad BD = DA; \quad (641)$$

$$\therefore BD + DC = DA + DC;$$

$$\therefore BC > AC. \quad \text{Q.E.D.} \quad (625)$$

651. COR. Conversely, if, in triangle ABC , BC is greater than AC , then angle A is greater than angle B .

For these angles cannot be equal, since BC and AC are not equal (Hyp.).

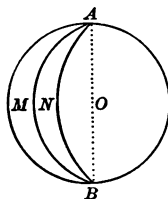
Nor can $\angle A$ be less than $\angle B$, for then BC would be less than AC (650); hence $\angle A$ must be greater than $\angle B$.

652. DEFINITION. A *lune* is the part of the surface of a sphere that is included between two semicircumferences of great circles; as $AMBNA$.

The *angle* of the lune is that between the semicircumferences that form its sides.

653. COR. *Lunes on the same sphere having equal angles are equal.*

For they evidently can be made to coincide.



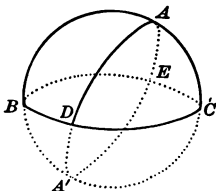
654. DEFINITION. A *spherical ungula* or *wedge* is the part of a sphere bounded by a lune as base, and by the planes of its sides; as $AONMB$.

The *angle* of the ungula is the same as the angle of its base; the diameter AOB is called the *edge* of the ungula.



PROPOSITION XXII. THEOREM.

655. *If two arcs of great circles intersect on the surface of a hemisphere, the sum of the two opposite triangles thus formed is equivalent to a lune whose angle is equal to that formed by the arcs.*



Given: On the hemisphere $A-BDCE$, two arcs of great circles, BAC and DAE , intersecting at A , and forming the $\triangle ABD$, ACE ;

To Prove: $\triangle ABD + \triangle ACE \approx$ a lune whose $\angle = \angle CAE$.

Produce arcs AC , AE , till they meet at A' .

Since BAC = a semicircumf. = ACA' , (Hyp.)

$$BAC - AC = ACA' - AC; \quad (\text{Ax. 3})$$

$$\text{i.e., } AB = A'C.$$

In the same way it may be shown that

$$AD = A'E \text{ and } BD = EC;$$

$\therefore \triangle ABD$, $A'CE$, are mutually equilateral;

$$\therefore \triangle ABD \cong \triangle A'CE; \quad (645, 649)$$

$$\therefore \triangle ABD + \triangle ACE \cong \triangle A'CE + \triangle ACE;$$

$$\text{i.e., } \triangle ABD + \triangle ACE \cong \text{lune } ACA'E,$$

whose \angle is $\angle CAE$.

Q.E.D.

656. DEFINITION. Just as the angle which is the ninetyeth part of a right angle is called a *degree*, so the trirectangular triangle which is the ninetyeth part of a trirectangular triangle, and has for base the arc that measures one degree, is called a *spherical degree*. Denoting by T the surface of a trirectangular triangle, the surface of the sphere = $8T = 720$ spherical degrees. It is also obvious that the lune whose angle is one degree will contain two spherical degrees.

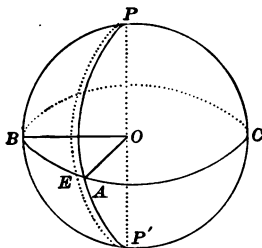
It is important to keep in mind the distinction between the three different senses in which the term *degree* is employed. A degree of angular measure is the 90th part of a right angle, a relative position of two straight lines; an arc degree is a *line*, the 90th part of a quadrant or 360th part of a circumference; a spherical degree is a *surface*, the 90th part of a trirectangular triangle or the 360th part of the surface of a hemisphere.

EXERCISE 803. If a spherical triangle has one right angle, the sum of the acute angles is greater than a right angle.

804. Lunes with equal angles on unequal spheres are similar.

PROPOSITION XXIII. THEOREM.

657. *A lune is to the surface of the sphere as the angle of the lune is to four right angles.*



Given: A lune L upon a sphere whose surface is S , and AOB , the angle of the lune, whose poles are P, P' ;

To Prove: $L : S = \text{angle } AOB : 4 \text{ right angles.}$

1°. When arc AB and circumference ABC are commensurable.

Let arc AE be a common measure of arc AB and the circumference, so that AE can be laid off 9 times on AB and 73 times on ABC .

Suppose arcs of great circles to be passed through the points of division and the poles P, P' . The lunes whose angles are measured by the equal arcs are equal (653), and each is contained 9 times in L and 73 times in S .

$$\text{Since } \angle AOB : 4 \text{ rt. } \angle = 9 : 73, \quad (\text{Hyp.})$$

$$\text{and } L : S = 9 : 73, \quad (\text{Const.})$$

$$L : S = \angle AOB : 4 \text{ rt. } \angle. \quad \text{Q.E.D. } (232''')$$

2°. When arc AB and the circumference are incommensurable, we can prove by the method of limits (as in Prop. II., Book IV.) that always

$$L : S = \angle AOB : 4 \text{ rt. } \angle. \quad \text{Q.E.D.}$$

658. COR. 1. *On the same sphere, lunes are to each other as their angles.*

659. COR. 2. *Denoting by A the number of degrees in the angle of a lune,*

$$\text{since } L : 8 T = A : 360,$$

$$L = T \times \frac{A}{45}.$$

660. COR. 3. *The spherical ungula $AB - PP'$: sphere $= \angle AOB : 360$.*

For unguulas are equal if their lunes are equal, since they can be made to coincide. Hence in equal spheres, unguulas are to each other as their lunes; or, employing v to denote the volume of the sphere, and U that of the ungula, we have

$$U : v = A : 360;$$

$$U = v \times \frac{A}{45}.$$

661. SCHOLIUM. In the above formulas, the symbols L , T , U , v , denote only relative values. In order to obtain the absolute value of L or U , we must know that of T or v , which, again, depend upon the radius of the sphere; in what manner will be seen further on.

EXERCISE 805. The angle of a lune is 36° . What fraction is the lune of the surface of its sphere?

806. A lune comprises one hundredth part of the surface of a sphere. What is the angle of that lune?

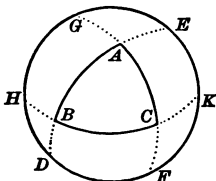
807. A trirectangular triangle is to a lune on the same sphere as 12 is to 5. What is the angle of the lune, and what fraction is it of the whole spherical surface?

808. The dihedral angle formed by the plane faces of a spherical wedge is 25° . What fraction is that wedge of the whole sphere?

809. In order that a spherical wedge shall be $\frac{1}{8}$ th of its sphere, what must be the angle of its base?

PROPOSITION XXIV. THEOREM.

662. *A spherical triangle is equivalent to as many spherical degrees as there are angular degrees in its spherical excess.*



Given: A spherical $\triangle ABC$, whose spherical excess is E degrees;

To Prove: Triangle ABC is equivalent to E spherical degrees.

Produce the sides to meet DKH , a great circle described about ABC .

Since $\triangle ADF + \triangle AEG \approx$ a lune whose $\angle = \angle A$, (655)

$\triangle ADF + \triangle AEG \approx 2 A$ spherical degrees. (656)

In the same way it may be proved that

$\triangle BEK + \triangle BDH \approx 2 B$ spher. deg.,

$\triangle CGH + \triangle CFK \approx 2 C$ spher. deg.

Now the sum of these six triangles exceeds the surface of the hemisphere, or 360 spher. deg., by twice $\triangle ABC$;

$\therefore 360$ spher. deg. $+ 2 \triangle ABC \approx 2(A + B + C)$ spher. deg.;

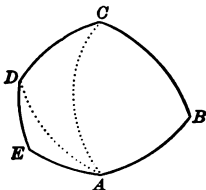
$\therefore \triangle ABC \approx (A + B + C - 180)$ spher. deg.;

i.e., $\triangle ABC \approx E$ spher. deg. Q.E.D. (636)

663. SCHOLIUM. E , it must be remembered, is here employed as a numerical measure, an abstract number. If, for example, the angles of the spherical triangle ABC are 73° , 87° , and 100° , respectively, then $E = 73 + 87 + 100 - 180 = 80$. Hence we know that $\triangle ABC \approx 80$ spherical degrees $\approx \frac{80}{360} T \approx \frac{1}{4.5}$ of the surface of the sphere.

PROPOSITION XXV. THEOREM.

664. *Any spherical polygon is equivalent to as many spherical degrees as there are angular degrees in its spherical excess.*



Given: A spherical polygon $ABCDE$, whose spherical excess is E degrees;

To Prove: Polygon $ABCDE$ is equivalent to E spherical degrees.

Through any vertex A and the opposite vertices, describe arcs of great circles, AC , AD , dividing the polygon into spherical triangles.

Now the area of each triangle is equal to as many spherical degrees as there are angular degrees in its spherical excess (661). Hence the area of the polygon is equal to as many spherical degrees as there are degrees in the sum of the spherical excesses of the triangles; that is, as many degrees as there are in the spherical excess of the polygon; (637)

\therefore polygon $ABCDE \approx E$ spherical degrees. Q.E.D.

665. COR. *The area of any spherical polygon is to the surface of the sphere as E is to 720.*

For (636, 637) E being the spherical excess of any polygon P , whether of three or more sides, and S the surface of the sphere,

$$P : S = E : 720.$$

EXERCISE 810. If the angles A , B , C , D , E , of the spherical pentagon $ABCDE$ are 140° , 90° , 93° , 120° , and 117° , respectively, what part of the spherical surface is $ABCDE$?

EXERCISES.

QUESTIONS.

811. What is the locus of all the points in space that are at a given distance from a given straight line ?

812. The cylinder may be regarded as the limiting form of what solid ?

813. What is the locus of all the straight lines in space that make a given angle with a given straight line at a given point ?

814. The cone may be regarded as the limiting form of what solid ?

815. What is the locus of all the points in space that are at a given distance from a given point ?

816. Two plane triangles that are mutually equiangular are not necessarily equilateral ; but spherical triangles on equal spheres, if mutually equiangular, are also mutually equilateral. Why so ?

817. If straight lines be drawn from any point in a spherical surface to the extremities of a diameter, what angle will those lines contain ?

818. The plane that is tangent to a sphere at a given point is the locus of what lines ?

819. What is the locus of all the points in space that have their distances from two given parallel lines in a given ratio ?

820. What is the locus of all the points in space such that the distances of each from a given straight line and a given point in that line have a given ratio ?

821. What is the locus of all the points in space at a given distance from a given plane ?

822. What is the locus of all the points in space at a given distance from a given spherical surface ? Under what circumstances will the locus consist of one surface only ?

823. What is the locus of all the points in space at a given distance from a given circular cylindrical surface ? Under what circumstances will the locus consist of one surface only ?

824. What is the locus of all the points in space at a given distance from a given circular conical surface ?

THEOREMS.

825. The locus of all the points such that lines drawn from it to the extremities of a given straight line form a right angle, is a spherical surface.

826. If any number of lines in space pass through a given point, the feet of the perpendiculars from any other point to these lines lie upon a spherical surface.

827. If any number of lines in a plane pass through a point, the feet of the perpendiculars to those lines from any point without the plane lie in a circle.

828. If from a point on the surface of a sphere as pole, with a polar distance equal to one third the chord of a quadrant, a circle be described, the radius of this circle will be one half the radius of the sphere.

829. Any lune is to a trirectangular triangle as its angle is to half a right angle.

830. Spherical polygons on equal spheres are as their spherical excesses.

831. In any right spherical triangle, if one side be greater than a quadrant, there must be a second side greater than a quadrant.

832. In any right spherical triangle, a side less than a quadrant subtends an acute angle ; a side greater than a quadrant subtends an obtuse angle.

833. Two right spherical triangles are equal or symmetrical if the hypotenuse and an adjacent angle of the one are severally equal to the hypotenuse and an adjacent angle of the other.

834. Two right spherical triangles are equal or symmetrical if the hypotenuse and an arm of the one are severally equal to the hypotenuse and an arm of the other.

835. The bisector of the angle contained by arcs of great circles is the locus of all points within the angle and equidistant from its sides.

BOOK X.

MEASUREMENT OF THE THREE ROUND BODIES.



CYLINDERS.

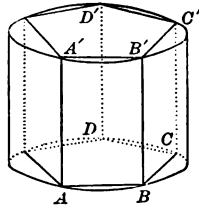
666. A prism is *inscribed* in a cylinder when its bases are inscribed in the bases of the cylinder, and its lateral edges are elements of the lateral surface of the cylinder.

667. A prism is *circumscribed* about a cylinder when its bases are circumscribed about the bases of the cylinder.

668. The *lateral area* of a cylinder is the area of its lateral surface.

PROPOSITION I. THEOREM.

669. *The lateral area of a cylinder of revolution is measured by the product of the circumference of its base by its altitude.*



Given: C , the circumference of the base of a cylinder of revolution $ABCD-C'$; H , its altitude; and S , its lateral surface;

To Prove: S is equal to $C \times H$.

Inscribe in the cylinder a regular prism $ABCD-C'$, whose bases are regular polygons inscribed in the bases of the cylinder.

If we denote by s the lateral surface of the prism, and by p the perimeter of each of the base polygons; then, H being also its altitude,

$$s = p \times H, \quad (514)$$

whatever be the number of lateral faces of the prism.

Let the number of lateral faces be indefinitely increased by continually doubling the number of sides of the base polygons.

Then as p has for limit C (392), and the lateral edges of the prism are, if indefinitely increased in number, the elements of the surface S ,

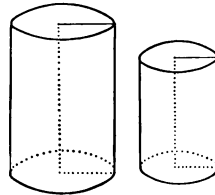
s has for limit S ;

$$\therefore S = C \times H. \quad \text{Q.E.D.} \quad (236)$$

670. COR. 1. *If the cylinder be generated by a rectangle whose sides are H and R , revolving about H , then H is the altitude of the cylinder and R the radius of the base. Hence the perimeter of the base is $2\pi \cdot R$ (396), and for the lateral area S we obtain the expression, $S = 2\pi \cdot R \cdot H$.*

671. COR. 2. *Since the area of each base is $\pi \cdot R^2$ (398), we obtain for T , the total area of a cylinder of revolution, the expression, $T = 2\pi \cdot R(H + R)$.*

672. DEFINITION. *Similar cylinders of revolution are generated by similar rectangles revolving about homologous sides.*



673. COR. 3. *The lateral areas, or the total areas, of similar cylinders of revolution, are as the squares of their altitudes or radii.*

For let s and s' denote the lateral areas of two similar cylinders of revolution; R and R' , the radii of their bases; H and H' , their altitudes; T and T' , their total areas;

then, since the generating rectangles are similar, (Hyp.)

$$H : H' = R : R' = H + H' : R + R'; \quad (246)$$

$$\therefore S : S' = 2\pi \cdot R \cdot H : 2\pi \cdot R' \cdot H' = H^2 : H'^2 = R^2 : R'^2,$$

$$\text{and } T : T' = 2\pi \cdot R(H + R) : 2\pi \cdot R'(H' + R') = H^2 : H'^2 = R^2 : R'^2.$$

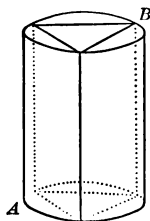
674. SCHOLIUM. The lateral area of any cylinder is equal to the product of the perimeter of a right section of the cylinder by an element of its surface.

This may be proved by a method similar to that employed in the proof of Prop. I., assuming that the bases of the cylinders, when not circular, are still the limits of inscribed polygons, the number of whose sides is indefinitely great.



PROPOSITION II. THEOREM.

675. *The volume of a cylinder of revolution is measured by the product of its base by its altitude.*



Given: V , the volume of a cylinder of revolution AB , whose base is B and altitude H ;

To Prove: V is equal to $B \times H$.

Let v' denote the volume of a regular inscribed prism of any number of faces; b' , its base; h will also be its altitude.

Now whatever be the number of sides of the prism,

$$V' = B' \times H. \quad (535)$$

But when the number of faces is indefinitely increased,

B' has for limit B , and V' has for limit V ;

$$\therefore V = B \times H. \quad \text{Q.E.D.} \quad (236)$$

676. COR. 1. *Let V be the volume of the cylinder, R the radius of its base, and H its altitude; then, since $B = \pi \cdot R^2$,*

$$V = \pi \cdot R^2 \cdot H.$$

677. COR. 2. *The volumes of similar cylinders of revolution are to each other as the cubes of their altitudes or radii.*

For if V and V' be the volumes of two similar cylinders of revolution, R and R' the radii of their bases, H and H' their altitudes;

since the generating rectangles are similar, (Hyp.)

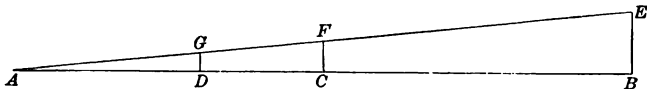
$$H : H' = R : R';$$

$$\therefore V : V' = \pi \cdot R^2 \cdot H : \pi \cdot R'^2 \cdot H' = H^3 : H'^3 = R^3 : R'^3.$$

678. SCHOLIUM. The volume of any cylinder is measured by the product of its base by its altitude.

This may be proved by the same method as that employed in Prop. II., making the assumption before referred to (674).

EXERCISE 836. Show that, by the following construction, two lines, M and N , can be found such that $M : N = 355 : 113$. Take $AB = 10$ units of any convenient length, and on it lay off $AC = 5$, and $AD = 3$.



Draw $BE \perp$ to AB and $= 1$. Join AE , and draw CF , DG , each \perp to AB , and meeting AE in F , G resp. Take $M = 3AB + AC + CF$, and $N = AB + BE + DG$. Then $M : N = 355 : 113$.

CONES.

679. A pyramid is *inscribed* in a cone when its base is inscribed in the base of the cone, and its vertex is that of the cone, the lateral edges of the pyramid thus being elements of the surface of the cone.

680. A pyramid is *circumscribed* about a cone when its base is circumscribed about the base of the cone, and its vertex is that of the cone.

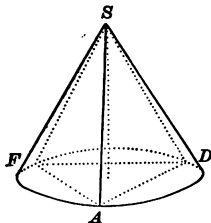
681. The *altitude* of a cone is the perpendicular distance from its vertex to its base.

682. The *slant height* of a cone of revolution is equal to the hypotenuse of the generating triangle.

683. The *lateral area* of a cone is the area of its lateral surface.

 PROPOSITION III. THEOREM.

684. *The lateral area of a cone of revolution is measured by the product of the circumference of its base by one half its slant height.*



Given: S , the lateral area; C , the circumference of the base; and L , the slant height of a cone of revolution $S-ADF$;

To Prove: S is equal to $\frac{1}{2} C \times L$.

Inscribe in the base any regular polygon ADF ; and upon this polygon as base construct the regular inscribed pyra-

mid $S-ADF$. If we denote by s the lateral area of the pyramid, by p the perimeter of its base, and by l its slant height,

$$s = \frac{1}{2} p \times l, \quad (549)$$

whatever be the number of lateral faces of the pyramid.

Conceive the number of lateral faces to be indefinitely increased by continually doubling the number of sides of the base polygon; then, since s , p , and l have for limits S , C , L , respectively,

$$S = \frac{1}{2} C \times L. \quad \text{Q.E.D.} \quad (536)$$

685. COR. 1. If R denote the radius of the base, then $C = 2\pi \cdot R$ (396), and $S = \frac{1}{2} (2\pi \cdot R \cdot L) = \pi \cdot R \cdot L$.

Also, since the area of the base $= \pi \cdot R^2$ (398), the total area, T , of the surface of a cone, is expressed by

$$T = \pi \cdot R \cdot L + \pi \cdot R^2 = \pi \cdot R(L + R).$$

686. DEFINITION. *Similar cones of revolution* are generated by similar right triangles revolving about homologous arms as axes.

687. COR. 2. *The lateral areas, or the total areas, of similar cones of revolution, are to each other as the squares of their altitudes, or as the squares of their radii.*

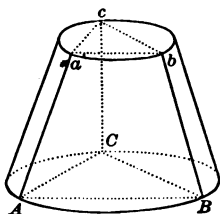
This may be proved as was Cor. 3 of Prop. I.

688. DEFINITION. A *truncated cone* is the portion of a cone intercepted between the base of the cone and a plane cutting its lateral surface.

689. DEFINITION. A *frustum* of a cone is a truncated cone that has the cutting plane parallel to the base. The section made by the cutting plane is the *upper base* of the frustum; the perpendicular distance between its bases is the *altitude* of the frustum; and the portion of the slant height of the cone that is intercepted between the bases is the *slant height* of the frustum.

PROPOSITION IV. THEOREM.

690. *The lateral area of a frustum of a cone of revolution is measured by the product of its slant height by half the sum of the circumferences of its bases.*



Given: S , the lateral surface, C and c , the circumferences of its bases, and L , the slant height of $ABC-c$, a frustum of a cone of revolution;

To Prove: S is equal to $\frac{1}{2}L(C + c)$.

Inscribe in the frustum $ABC-c$ a frustum of a regular pyramid. If we denote its lateral surface by s , the perimeters of its upper and lower bases by p and P respectively, and its slant height by l ,

$$s = \frac{1}{2}l(P + p), \quad (550)$$

whatever be the number of lateral faces of the pyramid.

Conceive the number of lateral faces of the pyramidal frustum to be indefinitely increased by continually doubling the number of sides of its base polygons; then.

since s, p, P, l , have for limits S, c, C, L , respectively,

$$S = \frac{1}{2}L(C + c). \quad \text{Q.E.D.} \quad (236)$$

691. COR. *The lateral area of a frustum of a cone of revolution is measured by the product of its slant height by the circumference of a section equidistant from the bases.*

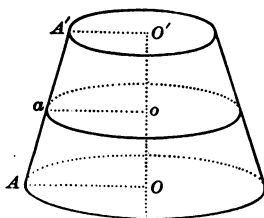
For if oa be the radius of a section equidistant from the bases, whose radii are OA and $O'A'$ respectively,

$$\text{since } oa = \frac{1}{2}(OA + O'A'), \quad (150)$$

$$2\pi \cdot oa = \frac{1}{2}(2\pi \cdot OA + 2\pi \cdot O'A'); \quad (151)$$

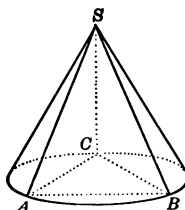
$$\therefore \text{circumf. } oa = \frac{1}{2}(\text{circumf. } OA + \text{circumf. } O'A'); \quad (152)$$

$$\therefore \text{circumf. } oa \times AA' = \frac{1}{2}(\text{circumf. } OA + \text{circumf. } O'A') \times AA'.$$



PROPOSITION V. THEOREM.

692. *The volume of a cone of revolution is measured by one third the product of its base by its altitude.*



Given: V , the volume, B , the base, H , the altitude, of a cone of revolution $S-ABC$;

To Prove: V is equal to $\frac{1}{3} B \times H$.

Inscribe in the cone a regular pyramid, $S-ABC$; then denoting its volume by V' , its base by B' , H being also its altitude,

$$V' = \frac{1}{3} B' \times H, \quad (555)$$

whatever be the number of lateral faces of the pyramid.

Conceive the number of lateral faces of the pyramid to be indefinitely increased ;

since V' and B' have for limits V and B resp.,

$$V = \frac{1}{3} B \times H. \quad \text{Q.E.D.} \quad (236)$$

693. COR. 1. *If R denote the radius of the base ; then, since*
 $B = \pi \cdot R^2,$ (398)

$$V = \frac{1}{3} \pi \cdot R^2 \cdot H.$$

694. COR. 2. *Similar cones of revolution are to each other as the cubes of their altitudes, or as the cubes of their radii.*

For if R and R' are the radii of two similar cones of revolution, H and H' their altitudes, V and V' their volumes,

since the generating triangles are similar, (Hyp.)

$$H : H' = R : R' ;$$

$$\therefore V : V' = \frac{1}{3} \pi \cdot R^2 \cdot H : \frac{1}{3} \pi \cdot R'^2 \cdot H' = H^3 : H'^3 = R^3 : R'^3.$$

695. SCHOLIUM. The volume of any cone is measured by one third the product of its base by its altitude.

This may be proved by the same method as that employed in the proof of Prop. V., making the assumption before referred to (674).

EXERCISE 837. If, by the method of Exercise 836, a straight line be found approximately equal to the circumference of a circle one yard in diameter, by what fraction of an inch will the line be too great, taking only two significant figures ?

838. The diameters of the bases of a frustum of a cone are 10 in. and 8 in. respectively, and its slant height is 12 in. Find its lateral area.

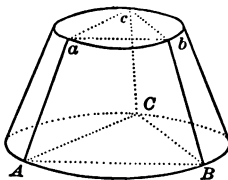
839. Find the area of a section of that same cone equidistant from its bases.

840. Find the volume of a cone of revolution the radius of its base being 10 in. and its altitude 20 in.

841. What is the altitude of a similar cone of twice the volume ?

PROPOSITION VI. THEOREM.

696. *The volume of a frustum of a cone of revolution is measured by one third the product of its altitude by the sum of the bases of the frustum and a mean proportional between those bases.*



Given: V , the volume, B and B' , the bases, and H , the altitude, of a frustum $ABC-c$;

To Prove: V is equal to $\frac{1}{3} H (B + B' + \sqrt{B \cdot B'})$.

Inscribe in the frustum a frustum, $ABC-c$, of a regular pyramid; then, denoting its volume by v , its bases by b and b' , H being its altitude,

$$v = \frac{1}{3} H (b + b' + \sqrt{b \cdot b'}), \quad (559)$$

whatever be the number of the lateral faces of the frustum.

Conceive the number of lateral faces of the inscribed frustum to be indefinitely increased;

since v , b , and b' , have for limits V , B , and B' , resp.,

$$V = \frac{1}{3} H (B + B' + \sqrt{B \cdot B'}). \quad \text{Q.E.D.} \quad (236)$$

697. COR. *If R and R' denote the radii of the bases of the frustum, as*

$$B = \pi \cdot R^2, \quad B' = \pi \cdot R'^2, \quad \text{and} \quad \sqrt{B \cdot B'} = \pi \cdot R \cdot R',$$

$$V = \frac{1}{3} \pi \cdot H (R^2 + R'^2 + R \cdot R').$$

698. SCHOLIUM. The volume of a frustum of any cone is measured by one third the product of its altitude, etc.

The same remark applies here as in Arts. 674 and 695.

EXERCISES.

NUMERICAL.

842. Find the lateral area and the total area of a cylinder of revolution whose altitude is 18 in. and the diameter of its bases 12 in.

843. What is the volume of the same cylinder ?

844. The total area of a cylinder of revolution is 700 sq. in., its altitude is 14 in. What is the diameter of a base ?

845. What should be the altitude of a cylinder of revolution, the diameter of which is 5 in., so that the lateral area shall be a square foot ?

846. What should be the diameter of a cylinder of revolution whose altitude is 10 in., so that its total area shall be 500 sq. in. ?

847. The altitude of a cylinder of revolution is three times its diameter ; the total area is 1200 sq. in. Find the altitude and diameter.

848. The altitudes of two similar cylinders of revolution are as 7 to 5. What is the ratio of their total areas ? Of their volumes ?

849. What formula expresses the total area of a cylinder of revolution whose altitude and radius are equal ?

850. What formula expresses the volume of the same cylinder ?

851. What is the ratio of the volume of the same cylinder to the volume of a cube having the same altitude ?

852. Find the lateral area and the total area of a cone of revolution whose altitude is 15 in., and the diameter of whose base is 12 in.

853. Find the volume of the same cone.

854. The total area of a cone of revolution is 400 sq. in. ; its altitude is 10 in. What is the diameter of its base ?

855. What should be the altitude of a cone of revolution whose base has a diameter of 10 in., so that the lateral area may be a square foot ?

856. What should be the radius of the base of a cone of revolution whose altitude is 10 in., so that its total area shall be 100 sq. in. ?

857. The altitude of a cone of revolution is four times the radius of its base ; the lateral area is 500 sq. in. Find the radius and altitude.

858. The altitudes of two similar cones of revolution are as 11 to 8. What is the ratio of their total areas? Of their volumes?

859. What formula expresses the total area of a cone of revolution whose altitude is equal to the radius of its base?

860. What formula expresses the volume of the same cone?

861. What is the ratio of the volume of the same cone to the volume of a regular tetrahedron having the same altitude?

862. What should be the altitude of such a cone, that its lateral area may be 100 sq. in.?

863. What should be the altitude of such a cone, that its volume may be 1000 cu. in.?

864. What is the lateral area and the total area of a frustum of a cone of revolution whose altitude is 20 in., and the diameters of whose bases are 6 in. and 14 in. respectively?

865. What is the volume of the same frustum?

866. The diameters of the bases of a frustum of a cone of revolution are 10 in. and 16 in. respectively; its volume is 575 cu. in. What is its altitude?

867. How far from the base must a cone, whose altitude is 16 in., be cut by a plane so that the frustum shall be equivalent to one half the cone?

868. What is the ratio of the lateral surfaces of a right circular cylinder and a right circular cone of the same base and altitude, if the altitude is three times the radius of the base?

869. The diameter of a right circular cylinder is 10 ft., and its altitude 7 ft. What is the side of an equivalent cube?

870. The altitude of a cone of revolution is 15 in., and the radius of its base 5 in. What should be the diameter of a cylinder of revolution having the same altitude and lateral area?

871. What should be the ratio of the exterior to the interior diameter of a hollow cylinder of revolution, so that it shall contain one half the volume of a solid cylinder of the same dimensions?

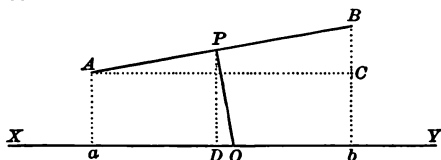
872. In order that a cylindrical tank with a depth of 12 ft. may contain 2000 gal., what should be its diameter?

873. How many cubic inches of iron would be required to make that tank, its walls being one third of an inch thick?

SPHERES.

PROPOSITION VII. THEOREM.

699. *The area of the surface generated by a straight line revolving about an axis in its plane, is measured by the product of the projection of that line upon the axis by the circumference of the circle whose radius is the perpendicular from the axis to the mid point of the line.*



Given: ab , the projection upon XY of AB revolving about XY , and $OP \perp$ to AB at its mid point, and meeting XY in O ;

To Prove: Area generated by AB is equal to $ab \times 2 \pi \cdot OP$.

Draw $PD \perp$ to XY , and $AC \parallel$ to XY .

Since the surface generated by AB is the lateral surface of a frustum of a cone, (576)

$$\text{area gen. by } AB = AB \times 2 \pi \cdot PD. \quad (689)$$

$$\text{Now } \triangle ABC \text{ is similar to } \triangle POD; \quad (292)$$

$$\therefore AB : OP = AC : PD; \quad (285)$$

$$\therefore AB \times PD = AC \times OP = ab \times OP; \quad (237)$$

$$\therefore AB \times 2 \pi \cdot PD = ab \times 2 \pi \cdot OP;$$

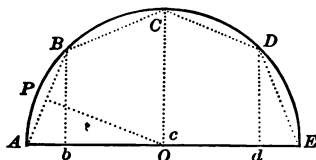
$$\text{i.e., area generated by } AB = ab \times 2 \pi \cdot OP. \quad \text{Q.E.D.}$$

If AB meets XY , the surface generated is still a conical surface whose area $= ab \times 2 \pi \cdot OP$, as follows from Prop. III.

If AB is parallel to XY , the surface generated is a cylindrical surface whose area $= ab \times 2 \pi \cdot OP$, as follows from Prop. I.

PROPOSITION VIII. THEOREM.

700. *The area of the surface of a sphere is measured by the product of its diameter by the circumference of a great circle.*



Given: S , the surface of a sphere generated by the revolution of the semicircle $ABCE$ about the diameter AOE , where OA equals R ;

To Prove: Area of S is equal to $AE \times 2\pi \cdot R$.

Inscribe in the semicircle a regular semipolygon $AB \dots E$, of any number of sides, and draw Bb, Cc, Dd, \perp to AE .

From O draw $OP \perp$ to AB . Then OP bisects AB (172), and is equal to each of the \perp drawn from O to the equal chords BC, CD, DE (182).

$$\left. \begin{aligned} \text{Now area } AB &= Ab \times 2\pi \cdot OP, \\ \text{area } BC &= bc \times 2\pi \cdot OP, \\ \text{area } CD &= cd \times 2\pi \cdot OP, \text{ etc.;} \end{aligned} \right\} \quad (699)$$

\therefore if S' denote the surface generated by the semipolygon,

$$S' = (Ab + bc + cd + dE) \times 2\pi \cdot OP = AE \times 2\pi \cdot OP.$$

Conceive the number of sides of the semipolygon to be indefinitely increased. Then, as OP has for limit R , the semipolygon for limit the semicircle, and S' for limit S ,

$$S = AE \times 2\pi \cdot R. \quad \text{Q.E.D.} \quad (236)$$

701. COR. 1. *The surface of a sphere is equivalent to four great circles.*

$$\text{For } S = 2R \times 2\pi \cdot R = 4\pi \cdot R^2, \quad (700)$$

$$\text{and } \pi \cdot R^2 \text{ is the area of a great circle.} \quad (398)$$

702. COR. 2. *The areas of the surfaces of two spheres are to each other as the squares of their radii or diameters.*

703. DEFINITIONS. A *zone* is a portion of the surface of a sphere included between two parallel planes. The *altitude* of the zone is the perpendicular distance between the parallel planes. The *bases* of the zone are the circumferences of the bounding circles. The zone is called a *zone of one base*, if one of the parallel planes is tangent to the sphere; that is, a zone of one base is the surface cut off by a plane.

704. COR. 1. *The area of a zone is measured by the product of its altitude by the circumference of a great circle.*

For (see diagram for Prop. VIII.) the area of the zone generated by the revolution of the arc $BC = bc \times 2\pi \cdot R$.

705. COR. 2. *Zones on the same sphere, or on equal spheres, are to each other as their altitudes.*

706. COR. 3. *The area of a zone of one base is measured by the area of the circle whose radius is the chord of the generating arc.*

For the arc AB generates a zone of one base whose area is

$$Ab \times 2\pi \cdot R = \pi \cdot Ab \times AE = \pi \cdot \overline{AB}^2, \quad (398)$$

(since $AB \times AE = \overline{AB}^2$.)

EXERCISE 874. The diameter of a sphere being 1 ft., what is the area of a great circle of that sphere, and of the sphere itself?

875. If the area of its surface is 400 sq. in., what is the diameter of the sphere?

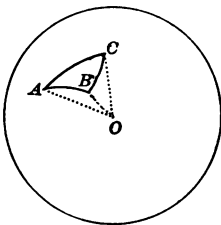
876. What is the ratio of the surfaces of two spheres whose radii are 10 in. and 12 in. respectively?

877. What is the area of a zone of a sphere 12 in. in diameter, the altitude of the zone being 3 in.?

878. What fraction of the diameter of a sphere should the altitude of a zone be so as to contain $\frac{1}{n}$ th of the surface?

PROPOSITION IX. THEOREM.

707. *The volume of a sphere is measured by one third the product of its surface by its radius.*



Given: V , the volume, and R , the radius, of the sphere whose surface is S ;

To Prove: V is equal to $\frac{1}{3} S \times R$.

Conceive a great number of points to be taken on the surface of the sphere, and join them, two and two, by arcs of great circles, so as to form on the surface a network of triangles such as ABC .

The planes of these arcs will, by their intersections, form the lateral faces of triangular pyramids, such as $O-ABC$, having their vertices at O , and their bases plane triangles whose vertices coincide with those of the spherical triangles.

Since the volume of each pyramid is equal to one third the product of its base and altitude (555), if we denote by V' the sum of these volumes, by b, b', b'', \dots the bases, and by h, h', h'', \dots the altitudes, we find

$$V' = \frac{1}{3} (b \cdot h + b' \cdot h' + b'' \cdot h'' + \dots).$$

Conceive the number of triangles to be indefinitely increased; then, as V' has for limit V , the sum of the bases has for limit S , and each altitude has for limit R ,

$$V = \frac{1}{3} S \cdot R. \quad \text{Q.E.D.} \quad (236)$$

708. COR. 1. Since $S = 4\pi \cdot R^2$, (700)

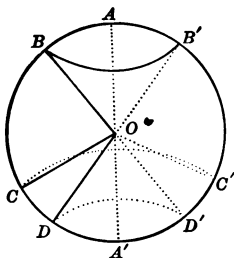
$$V = \frac{1}{3} \times 4\pi \cdot R^2 \cdot R = \frac{4}{3}\pi \cdot R^3 = \frac{1}{6}\pi \cdot D^3.$$

709. COR. 2. The volumes of spheres are to each other as the cubes of their radii or diameters.

For since $V = \frac{4}{3}\pi \cdot R^3 = \frac{1}{6}\pi D^3$, and $V' = \frac{4}{3}\pi R'^3 = \frac{1}{6}\pi \cdot D'^3$,

$$V : V' = R^3 : R'^3 = D^3 : D'^3.$$

710. DEFINITIONS. A *spherical sector* is a portion of a sphere generated by a sector of the semicircle that generates the sphere. Thus the revolution of the sector AOB generates the spherical sector $O-ABB'$. The revolution of the sector OCD generates the spherical sector $O-CDD'C'$. The base of the spherical sector is the zone generated by the circular sector.



711. COR. 1. The volume of a spherical sector is measured by one third the product of the zone that forms its base, by the radius of the sphere.

In the case of the spherical sector $O-ABB'$, it is manifest that whatever part the base BB' is of the surface of the sphere, that same part is $O-ABB'$ of the sphere. In the case of $O-CDD'C'$, we see that this sector consists of the difference of the spherical sectors $O-CA'C'$ and $O-DA'D'$, and its base consists of the difference of the portions of the surface of the sphere that are cut off by CC' and DD' respectively. If V denote the volume of the sector, Z the area of the zone, and H its altitude, then

$$V = \frac{1}{3} Z \cdot R = \frac{2}{3}\pi \cdot R^2 \cdot H.$$

712. COR. 2. The volumes of spherical sectors on equal spheres are as the zones that form their bases, or as their altitudes.

For since $V = \frac{1}{3} Z \cdot R = \frac{2}{3} \pi \cdot R^2 \cdot H$, and $V' = \frac{1}{3} Z' \cdot R' = \frac{2}{3} \pi \cdot R'^2 \cdot H$,

$$V : V' = Z : Z' = H : H'.$$

713. DEFINITION. A *spherical pyramid* is a solid bounded by a spherical polygon as *base*, whose *vertex* is the center of the sphere, and whose *sides* are the sides of the polyhedral angle formed by the planes of the sides of the polygon.

714. COR. 1. *The volume of a spherical pyramid is measured by one third the product of its base by the spherical radius.*

For it is evident that, whatever portion the base of the pyramid is of the surface of the sphere, the same portion will the volume of the pyramid be of the volume of the sphere.

715. COR. 2. *Spherical pyramids on equal spheres are to each other as their bases.*

716. DEFINITIONS. A *spherical segment* is a portion of a sphere included between two parallel planes. The *altitude* of the segment is the perpendicular distance between the parallel planes; the *bases* are the sections of the sphere made by those planes. If one of the bounding planes is tangent to the sphere, the segment is said to be a *segment of one base*.

EXERCISE 879. What is the volume of a sphere one foot in diameter?

880. Find the diameter of a sphere whose volume is one cubic foot.

881. Find the surface of a sphere whose volume is one cubic foot.

882. The diameters of two spheres are 10 in. and 12 in. respectively. Find the ratio (1) of their surfaces; (2) of their volumes.

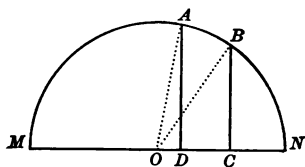
883. One sphere has twice the volume of another. Find the ratio of the radius of the first to the radius of the second.

884. The altitude of the base of a spherical sector of a sphere 10 in. in diameter is 2 in.; find the volume of the sector.

885. A spherical pyramid has for base a trirectangular triangle. What fraction is the pyramid of the sphere?

PROPOSITION X. THEOREM.

717. *The volume of a spherical segment is measured by one half the product of the sum of its bases by its altitude, plus the volume of a sphere of which that altitude is the diameter.*



Given: V , the volume of the segment generated by the revolution about MON of $ABCD$, where $AD = r$, $BC = r'$, and $CD = OC - OD = h$;

To Prove: $V = \frac{1}{2} h (\pi r^2 + \pi r'^2) + \frac{1}{6} \pi h^3$.

Join OA and OB . The solid generated by $ABCD$ consists of the spherical sector generated by OAB , plus the cone generated by OBC , minus the cone generated by OAD ;

$$\therefore V = \frac{2}{3} \pi h \cdot R^2 + \frac{1}{3} \pi \cdot \overline{BC}^2 \cdot OC - \frac{1}{3} \pi \cdot \overline{AD}^2 \cdot OD \quad (711; 692)$$

$$= \frac{1}{3} \pi \{ 2 R^2 \cdot h + (R^2 - \overline{OC}^2) OC - (R^2 - \overline{OD}^2) OD \} \quad (347)$$

$$= \frac{1}{3} \pi \{ 2 R^2 \cdot h + R^2 (OC - OD) - (\overline{OC}^3 - \overline{OD}^3) \}$$

$$= \frac{1}{3} \pi h \{ 3 R^2 - (\overline{OC}^2 + OC \cdot OD + \overline{OD}^2) \}.$$

But since $(OC - OD)^2 = h^2$, (Hyp.)

$$\overline{OC}^2 + OC \cdot OD + \overline{OD}^2 = \frac{3}{2} (\overline{OC}^2 + \overline{OD}^2) - \frac{h^2}{2}$$

$$= 3 R^2 - \frac{3}{2} (r^2 + r'^2) - \frac{h^2}{2}.$$

$$\therefore V = \frac{1}{3} \pi h \left\{ \frac{3}{2} (r^2 + r'^2) + \frac{h^2}{2} \right\}$$

$$= \frac{1}{2} h (\pi r^2 + \pi r'^2) + \frac{1}{6} \pi h^3.$$

Q.E.D.

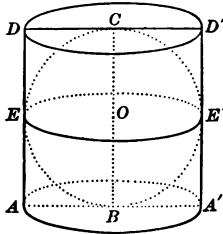
718. COR. *The volume of a spherical segment of one base is measured by one half the volume of the cylinder having the same base and altitude, plus the volume of the sphere having that altitude for diameter.*

For when $r' = 0$, i.e., when the segment has but one base, then

$$V = \frac{1}{2} \pi r^2 h + \frac{1}{6} \pi h^3.$$

PROPOSITION XI. THEOREM.

719. *The total surface and the volume of a cylinder circumscribed about a sphere, are respectively to the surface and volume of the sphere as 3 is to 2.*



Given: A cylinder AD' , circumscribed about a sphere $BECE'$;

To Prove: $\text{Surf. } AD' : \text{surf. } BECE' = \text{vol. } AD' : \text{vol. } BECE' = 3 : 2.$

For we may suppose the sphere and the cylinder to be generated respectively by the revolution about BC of the semicircle BEC and the rectangle BD that is circumscribed about BEC . Then

$$\text{surf. } AD' = 2 \pi \cdot AB (AB + BC) = 6 \pi \cdot \overline{AB}^2; \quad (669)$$

$$\text{surf. } BECE' = 4 \pi \cdot \overline{AB}^2 = 4 \pi \cdot \overline{AB}^2; \quad (700)$$

$$\text{vol. } AD' = \pi \cdot \overline{AB}^2 \cdot BC = 2 \pi \cdot \overline{AB}^3; \quad (674)$$

$$\text{vol. } BECE' = \frac{1}{6} \pi \cdot \overline{BC}^3 = \frac{4}{3} \pi \cdot \overline{AB}^3. \quad (708)$$

Hence, making the necessary simplifications, we have
 $\text{surf. } AD' : \text{surf. } BECE' = \text{vol. } AD' : \text{vol. } BECE' = 3 : 2. \quad \text{Q.E.D.}$

This interesting theorem is known as the Theorem of Archimedes, it having been discovered by that celebrated geometer.

720. SCHOLIUM. If we have a cone having the same base and altitude as the cylinder circumscribed about the sphere, then, since the volume of such a cone is $\frac{1}{3}\pi \cdot AB \cdot BC = \frac{2}{3}\pi \cdot \overline{AB}^3$, we obtain the relation :

$$\text{cylinder} : \text{sphere} : \text{cone} = 3 : 2 : 1.$$

EXERCISES.

NUMERICAL.

886. Find the area of the surface of a sphere whose radius is $3\frac{1}{2}$ in.
887. The surface of a sphere is to be 100 sq. in. What radius should be taken ?
888. Two spheres have radii of 9 in. and 5 in. respectively. What is the ratio of the surfaces of those spheres ? Of their volumes ?
889. The areas of the surfaces of two spheres are as 125 to 27. What is the ratio of their diameters ? Of their volumes ?
890. Two parallel planes intersect a sphere of 18 in. radius at distances of 9 in. and 13 in., respectively, from the center. Find the area of the intercepted zone.
891. What is the volume of the spherical sector that has for base the zone just mentioned ?
892. Find the altitude of a zone whose area is 100 sq. in. on the surface of a sphere of 12 in. radius.
893. In the same sphere, what is the altitude of a zone that contains one fourth of the surface of the sphere ?
894. What is the volume of a sphere whose diameter is (1) 1 ft. ; (2) 18 in. ?

895. The surface of a sphere is 64 sq. in. Find its volume.
896. The volume of a sphere is 5 cu. ft. Find its diameter and surface.
897. Find the difference of the volumes of two spheres whose radii are 12 in. and 7 in. respectively.
898. The volumes of two spheres are as 27 to 8. Find the ratio (1) of their diameter; (2) of their surfaces.
899. The radii of two spheres are as 4 to 5. Find the ratio (1) of their volumes; (2) of their surfaces.
900. In a sphere whose radius is 6 in., find the altitude of a zone whose area shall be that of a great circle.
901. The area of a zone forming the base of a spherical sector is 50 sq. in.; the radius of the sphere is 12 in. Find the altitude of the zone and the volume of the sector.
902. The volume of a spherical sector is 25 cu. in.; the diameter of the sphere is 14 in. Find the area of the zone that forms the base of the sector.
903. The altitude of a cylinder circumscribing a sphere is 5 in. Find the surface and volume of the sphere.
904. The volume of a sphere is one cubic foot. Find the surface of the circumscribing cylinder.
905. The surface and volume of a sphere are expressed by the same number. Find its diameter.
906. Find the volume of a sphere inscribed in a cube whose volume is 1331 cu. in.
907. Find the surface of a cube circumscribed about a sphere whose surface is 150 sq. in.
908. If a spherical shell have an exterior diameter of 12 in., what should be the thickness of its wall so that it may contain 696.9 cu. in.?
909. If an iron sphere, 6 in. in diameter, weigh n lbs., what will be the weight of an iron sphere whose diameter is 8 in.?
910. In a sphere 10 in. in diameter, the radius of the lower base of a spherical segment is 8 in. Find the volume of the segment, its altitude being 2 in.

THEOREMS.

911. The lateral area of a cylinder of revolution is equal to the area of a circle whose radius is a mean proportional between the altitude and diameter of the cylinder.

912. The lateral areas of the two cylinders generated by revolving a rectangle successively about each of its containing sides, are equal.

913. If the containing sides of the above rectangle are as m is to n , the total areas, and also the volumes, of the cylinders generated, will be as n is to m .

914. If the slant height of a cone of revolution is equal to the diameter of its base, its total area is to that of the inscribed sphere as 9 is to 4.

915. The arms of a right triangle are a and b . Find the area of the surface generated by revolving the triangle about its hypotenuse.

916. An equilateral triangle revolves about one of its altitudes. What is the ratio of the lateral surface of the generated cone to that of the sphere generated by the circle inscribed in the triangle?

917. An equilateral triangle revolves about one of its altitudes. Compare the volumes generated by the triangle, the inscribed circle, and the circumscribed circle respectively.

918. A circle of cardboard being given, what is the angle of the sector that must be cut from it so that with the remainder, a cone with a vertical angle of 90° may be formed?

919. If the diameter of a sphere be divided by a perpendicular plane in the ratio m to n , the zones thus formed will also be as m to n .

920. The volume of a cylinder of revolution is equal to one half the product of its lateral area by the radius of its base.

921. If the altitude of a cylinder of revolution is equal to the diameter of its base, its volume is equal to one third the product of its total area by the radius of its base.

922. The base of a cone is equal to a great circle of a sphere, and the altitude is equal to a diameter of the sphere. What is the ratio of their volumes?

923. The volume of a sphere is to that of the inscribed cube as π is to $2 \div \sqrt{3}$.

924. A sphere is to the circumscribed cube as π is to 6.

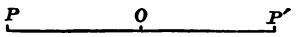
APPENDIX.



SYMMETRY.

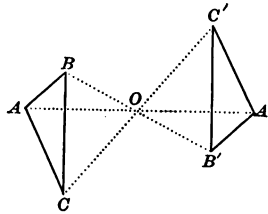
I. SYMMETRY WITH RESPECT TO A CENTER.

721. DEFINITION. Two points are said to be *symmetrical with respect to a third point*, if this point bisects the line joining the two points.

Thus the points P and P' are  symmetrical with respect to O , if the line PP' is bisected in O . The point O is then called the *center of symmetry*.

722. DEFINITION. Two figures are said to be *symmetrical with respect to a point*, called their *center of symmetry*, if every point in the one has its symmetrical point in the other.

Thus the figures ABC , $A'B'C'$, are symmetrical with respect to the center O , if every point in ABC has its symmetrical point in $A'B'C'$.



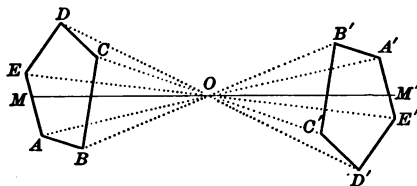
723. DEFINITION. In symmetrical figures, sides whose extremities are mutually symmetrical are said to be *homologous*. Thus AC is homologous to $A'C'$, since A is symmetrical with A' , and C with C' .

EXERCISE 925. The opposite vertices of a regular polygon of an even number of sides have a common center of symmetry.

926. The opposite vertices of a parallelopiped have a common center of symmetry.

PROPOSITION I. THEOREM.

724. *If two polygons are symmetrical with respect to a center, any two homologous sides are equal and parallel, and drawn in opposite directions.*



Given: Two polygons $AB \dots E$, $A'B' \dots E'$, symmetrical with respect to O ;

To Prove: AB , BC , etc., are resp. = and \parallel to $A'B'$, $B'C'$, etc.

Since $OA = OA'$, $OB = OB'$, (Hyp.)

and $\angle AOB = \angle A'OB'$, (50)

$\triangle AOB = \triangle A'OB'$; (66)

$\therefore AB = A'B'$, and $\angle OAB = \angle OA'B'$; (70)

$\therefore AB$ is \parallel to $A'B'$; (110)

also AB , $A'B'$, are drawn in opposite directions; (115)

and similarly for BC and $B'C'$, CD and $C'D'$, etc. Q.E.D.

725. COR. 1. *Any line MM' , intercepted between two homologous sides, AE , $A'E'$, and passing through O , is bisected in O .*

For since AE is \parallel to $A'E'$, the triangles AOM , $A'OM'$, are equiangular; and $OA = OA'$; $\therefore OM = OM'$ (70).

726. COR. 2. *If two polygons have their sides respectively equal and parallel, and drawn in opposite directions, they have a center of symmetry.*

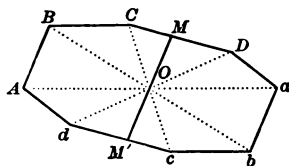
For if AB and $A'B'$ are equal and parallel, and drawn on opposite sides of AA' , BB' , then AA' , BB' , are diagonals of

what could be made a parallelogram (142); hence AA' , BB' , bisect each other in O (146).

727. SCHOLIUM. When two polygons are symmetrical with respect to a center, one can be made to coincide with the other by revolving it about the center through two right angles in their common plane.

728. DEFINITION. A figure is *symmetrical with respect to a point*, if every intercept that passes through the point is bisected there.

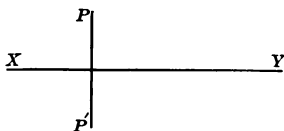
Thus the figure $AB \dots cd$ is symmetrical with regard to O , if every intercept, as MM' , that passes through O , is bisected in O .



II. SYMMETRY WITH RESPECT TO AN AXIS.

729. DEFINITION. Two points are said to be *symmetrical with respect to a straight line*, if this line bisects at right angles the straight line joining the two points.

Thus the points P and P' are symmetrical with respect to XY , if XY bisects PP' at right angles. The line XY is then called the *axis of symmetry* as regards P and P' .



EXERCISE 927. A circle is symmetrical with respect to what point?

928. A parallelogram is symmetrical with respect to what point?

929. A trapezium has no center of symmetry.

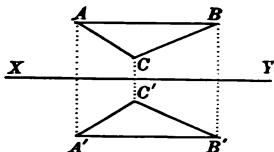
930. Every regular polygon of an even number of sides has a center of symmetry.

931. The axis of symmetry of the extremities of a chord is what line?

932. The axis of symmetry of opposite vertices of a square is what line?

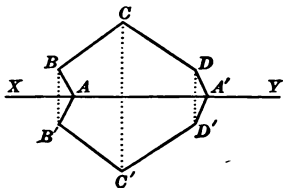
730. DEFINITION. Any two figures are said to be *symmetrical with respect to an axis*, if every point in the one has a point in the other symmetrical with respect to that axis.

Thus the figures ABC , $A'B'C'$, are symmetrical with respect to XY , if corresponding to every point in ABC there is a point in $A'B'C'$ symmetrical with respect to XY .



731. SCHOLIUM. It is obvious that, if the portion of the plane above XY be revolved about XY as an axis, till it coincides with the portion of the plane below XY , the figure ABC will coincide with $A'B'C'$, since the homologous points are at equal distances from XY .

732. DEFINITION. A plane figure is *symmetrical with respect to an axis*, if the axis divides the figure into two symmetrical figures. Thus the figure $AB \dots B'$ is symmetrical with regard to XY if its homologous points are symmetrical with respect to XY .



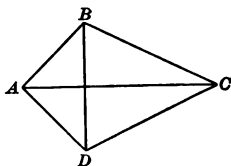
EXERCISE 933. How many axes of symmetry may a circle have? What common axis of symmetry have two circles?

934. An isosceles triangle is symmetrical with respect to which altitude?

935. An equilateral triangle has how many axes of symmetry?

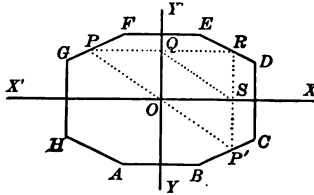
936. How many axes of symmetry may be drawn for (1) a square? (2) A rhombus? (3) A regular pentagon? (4) A regular hexagon? (5) A regular polygon of $2n$ sides? (6) Of $2n + 1$ sides?

937. In a quadrilateral $ABCD$, $AB = AD$, and $CB = CD$. Show that AC is an axis of symmetry, and is perpendicular to BD .



PROPOSITION II. THEOREM.

733. *If a figure is symmetrical with respect to two axes at right angles to each other, it is also symmetrical with respect to their intersection as center.*



Given: $AB \dots H$, symmetrical with respect to axes XX' , YY' , intersecting in O ;

To Prove: $AB \dots H$ is symmetrical with respect to O as center.

From P , any point in the perimeter of the figure, draw $PQR \perp$ to YY' , and through R draw $RSP' \perp$ to XX' .

Join OP , OP' , and QS .

Since $PQ = QR$, (Hyp.)

and $OS = QR$, (136)

$PQ = OS$. (Ax. 1)

Also PQ is \parallel to OS ; (106)

$\therefore OP$ is \parallel and $=$ to QS . (136)

In the same way it may be proved that

OP' is \parallel and $=$ to QS ;

$\therefore POP'$ is a straight line, and is bisected in O ;

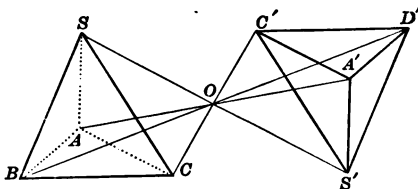
i.e., O is the center of symmetry of $AB \dots H$. Q.E.D.

734. SCHOLIUM. The axes XX' , YY' , evidently divide the figure into four equal parts. Any one of these parts may be made to coincide with either of the adjacent parts by revolving it about one of the axes, or may be made to coincide with the opposite part by revolving it, in the plane of the figure, through two right angles.

SYMMETRICAL POLYHEDRONS.

I. SYMMETRY WITH RESPECT TO A CENTER.

735. DEFINITION. Two polyhedrons are said to be *symmetrical with respect to a center*, when each vertex of the one has its symmetrical vertex on the other polyhedron.

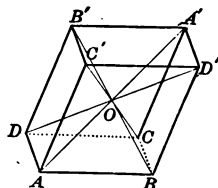


Thus in the polyhedrons $S-ABC$, $S'-A'B'C'$, if the lines joining the vertices A and A' , B and B' , etc., all pass through the same point O , and are bisected in that point, the polyhedrons are said to be symmetrical with respect to O , which is called their *center of symmetry*.

736. COR. If two polyhedrons are symmetrical with respect to a center, their homologous faces are severally equal, their dihedral angles are severally equal, their polyhedral angles are symmetrical, and the polyhedrons are equivalent.

737. DEFINITION. A polyhedron $ABCD-A'$ is said to be *symmetrical with respect to a center* O , if its vertices, taken two and two, are symmetrical with regard to O ; i.e., if AA' , BB' , etc., are each bisected in the same point O .

738. COR. A polyhedron, in order to have a center of symmetry, must have an even number of edges, must have its homologous edges equal and parallel, its homologous plane angles and dihedral angles equal, and its homologous polyhedral angles symmetrical.

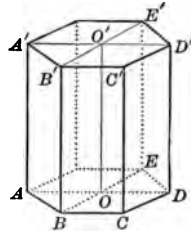


II. SYMMETRY WITH RESPECT TO AN AXIS.

739. DEFINITION. Two polyhedrons are said to be *symmetrical with respect to an axis*, when this axis is an axis of symmetry for the corresponding vertices of the two polyhedrons.

740. DEFINITION. A polyhedron is said to be *symmetrical with respect to an axis*, when this line is an axis of symmetry for the corresponding vertices of the polyhedron, taken two and two.

Thus the polyhedron $AB \dots E'$ is symmetrical with respect to the axis OO' when this axis is an axis of symmetry for each of the pairs of vertices, A and D , B' and E' , etc.



III. SYMMETRY WITH RESPECT TO A PLANE.

741. DEFINITION. Two points are said to be *symmetrical with respect to a plane*, when the plane is perpendicular to, and bisects, the straight line joining the two points.

EXERCISE 938. Every intercept between two opposite faces of a polyhedron having a center of symmetry, and passing through that center, is bisected there.

939. Every prism whose bases are polygons symmetrical with respect to a point, has a center of symmetry.

940. Where is the center of symmetry of a parallelepiped?

941. A right prism whose bases are symmetrical with respect to a center, has an axis of symmetry.

942. A rectangular parallelepiped has three axes of symmetry.

943. How many axes of symmetry has a cube?

944. A regular pyramid having an even number of lateral faces, has an axis of symmetry.

742. DEFINITION. Two polyhedrons are said to be *symmetrical with respect to a plane*, when this plane is a plane of symmetry for the corresponding vertices of the polyhedrons, taken two and two.

743. COR. *In order that two polyhedrons may be symmetrical with respect to a plane, their homologous faces must be severally equal, their dihedral angles must be equal, their polyhedral angles must be symmetrical, and the polyhedrons must be equivalent.*

744. DEFINITION. A polyhedron is said to be *symmetrical with respect to a plane*, when the plane divides it into two polyhedrons symmetrical with respect to that plane.



MAXIMA AND MINIMA.

745. DEFINITION. A magnitude is said to be a *maximum* or *minimum* according as it is the *greatest* or *least* of a given class.

Thus the diameter of a circle is a maximum among all inscribed straight lines; and, among all the straight lines drawn from a given point to a given straight line, the perpendicular is the minimum.

746. DEFINITION. *Isoperimetric figures* are those which have equal perimeters.

EXERCISE 945. How many planes of symmetry has any right prism ?

946. How many has a parallelopiped ?

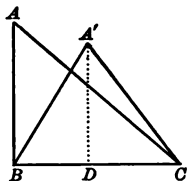
947. Has a cylinder a plane of symmetry ? A sphere ?

948. Show that, if a polyhedron has two planes of symmetry that are perpendicular to each other, their common intersection is an axis of symmetry.

949. Show that, if a polyhedron has three planes of symmetry, the point common to the three planes is a center of symmetry.

PROPOSITION I. THEOREM.

747. *Of all triangles having two given sides, that in which these sides are perpendicular to each other has the maximum area.*



Given : In triangles ABC , $A'BC$, AB equal to $A'B$, BC equal to BC , but only AB perpendicular to BC ;

To Prove : Triangle ABC is greater than triangle $A'BC$.

Draw $A'D \perp$ to BC .

Since $AB = A'B$, (Hyp.)

but $AB > A'D$, (93)

the altitude $AB >$ the altitude $A'D$;

$\therefore \triangle ABC > \triangle A'BC$. Q.E.D. (333)

EXERCISE 950. Two lines, whose lengths are a and b respectively, being given, find the length of the third line that will form with them the maximum triangle.

951. Of all triangles of given base and area, the isosceles has the greatest vertical angle.

952. Of all triangles of given base and vertical angle, the isosceles is the greatest.

953. Of all triangles of given altitude and vertical angle, the isosceles is the least.

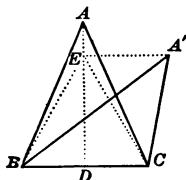
954. Of all triangles of given base and vertical angle, the isosceles has the greatest perimeter.

955. Divide a given arc into two parts such that the sum of the chords subtending them shall be a maximum.

749. COR. *Of all equivalent triangles, that which is equilateral has the least perimeter.*

PROPOSITION III. THEOREM.

750. *Of all isoperimetric triangles on the same base, that which is isosceles has the maximum area.*



Given: Two isoperimetric triangles ABC , $A'BC$, and AB equal to AC ;

To Prove: Triangle ABC is greater than triangle $A'BC$.

Draw $AD \perp$ to BC , and $A'E \parallel$ to BC , and meeting AD produced if necessary in E ; also join EB , EC .

Since $\triangle A'BC$, EBC , have the same altitude, (Const.)

$$\triangle A'BC \approx \triangle EBC. \quad (332)$$

But EBC is an isosceles \triangle ; (96)

$$\therefore A'B + A'C > EB + EC; \quad (748)$$

$$\therefore AB + AC > EB + EC, \quad (\text{Hyp.})$$

(since $AB + AC = A'B + A'C$;))

$$\therefore AB > EB; \quad (\text{Ax. 7})$$

$$\therefore AD > ED; \quad (99)$$

$$\therefore \triangle ABC > \triangle EBC \text{ or } \triangle A'BC. \quad \text{Q.E.D.} \quad (333)$$

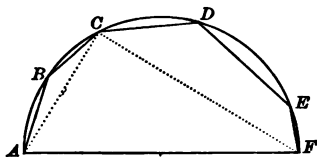
751. COR. *Of all isoperimetric triangles, that which is equilateral has the maximum area.*

EXERCISE 956. Find the area of the maximum triangle whose perimeter is n feet.

957. Of all triangles inscribed in a given circle, the equilateral has the greatest perimeter.

PROPOSITION IV. THEOREM.

752. *Of all polygons formed of sides all given but one, the maximum can be inscribed in a semicircle having the undetermined side as diameter.*



Given: $ABCDEF$, the maximum polygon, having the given sides AB , BC , CD , DE , EF , and an undetermined side AF ;

To Prove: $ABCDEF$ can be inscribed in a semicircle.

Join any vertex, as C , with A and F . Then the triangle ACF must be the maximum of all triangles formed with the given sides AC and CF .

For otherwise, by changing $\angle ACF$, leaving AC and CF unchanged, we could increase $\triangle ACF$, leaving the rest of the polygon unchanged.

That is, the whole polygon $AB \dots F$ would be increased, which it cannot be, since $AB \dots F$ is a maximum.

Hence $\triangle ACF$ must be the maximum triangle formed with the given sides AC , CF ;

$$\therefore \angle ACF \text{ is a rt. } \angle, \quad (747)$$

(since otherwise $\triangle ACF$ would not be a maximum;)

$\therefore C$ is on the semicircumf. whose diam. is AF ;

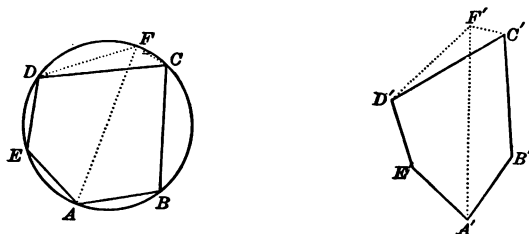
\therefore each vertex of $ABCDEF$ is on that semicircumf. Q.E.D.

EXERCISE 958. Of all parallelograms of a given base and area, the rectangle has the least perimeter.

959. Of all rectangles of given area, the square has the least perimeter.

PROPOSITION V. THEOREM.

753. *Of all polygons formed with given sides, that which can be inscribed in a circle is the maximum.*



Given: Two mutually equilateral polygons, $ABCDE$ and $A'B'C'D'E'$, of which only $ABCDE$ can be inscribed in a circle;

To Prove: $ABCDE$ is greater than $A'B'C'D'E'$.

Draw the diameter AF , and join FC , FD .

Upon $C'D'$ ($= CD$) construct $\triangle F'C'D' = \triangle FCD$, and join $A'F'$.

Since polygon $ABCF$ is inscribed in a semicircle,

$$\left. \begin{aligned} ABCF &> A'B'C'F'; \\ \text{similarly } AEDF &> A'E'D'F'; \end{aligned} \right\} \quad (752)$$

$$\therefore ABCFDE > A'B'C'F'D'E'; \quad (\text{Ax. 4})$$

$$\therefore ABCFDE - FCD > A'B'C'F'D'E' - F'C'D'; \quad (\text{Ax. 5})$$

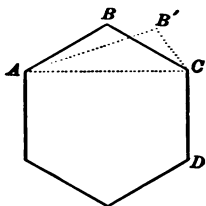
$$\text{i.e., } ABCDE > A'B'C'D'E'. \quad \text{Q.E.D.}$$

754. SCHOLIUM. The area of the inscribed polygon will be the same in whatever order the sides are arranged. For these sides are chords which cut off equal segments, in whatever order they occur; and the polygon is the difference between the circle and these segments.

EXERCISE 960. Of all rectangles that can be inscribed in a given circle, the greatest is a square.

PROPOSITION VI. THEOREM.

755. *Of isoperimetric polygons of the same number of sides, the maximum is a regular polygon.*



Given: $ABCD$, the maximum of isoperimetric polygons of n sides;

To Prove: $ABCD$ is a regular polygon.

The polygon $ABCD$ must be equilateral.

For if any two of its sides, BA , BC , were unequal, then upon AC as base we could construct an isosceles triangle $B'AC$, having the sum of its sides, $B'A$, $B'C$, equal to $BA + BC$. The triangle $B'AC$ would be greater than BAC (748), and therefore the polygon $AB'CD$ would be greater than the maximum polygon $ABCD$. But this is impossible; hence $ABCD$ must be equilateral. It can also be inscribed in a circle (753); hence it is a regular polygon (380).

EXERCISE 961. Of all triangles having the same vertical angle, and whose bases pass through a given point, that whose base is bisected by the given point is least.

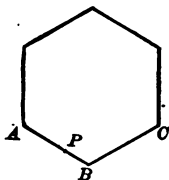
962. The rectangle contained by the segments of a line is a maximum when the segments are equal.

963. Through a given point within a given circle, draw the maximum and minimum chords that pass through that point.

964. From a given point without a given circle, draw the secant whose outer segment is a minimum. What about its inner segment?

PROPOSITION VII. THEOREM.

756. *Of isoperimetric regular polygons, that having the greatest number of sides is the maximum.*



Given: A regular polygon ABC , of n sides;

To Prove: ABC is less than an isoperimetric regular polygon of $n + 1$ sides.

In one of the sides, as AB , take any point P .

We may now regard the given polygon as an irregular polygon of $n + 1$ sides, in which the sides AP , PB , make with each other an angle equal to two right angles.

This irregular polygon is less than the regular polygon of the same perimeter and having $n + 1$ sides (755); that is, a regular polygon of n sides is less than the isoperimetric regular polygon of $n + 1$ sides. Q.E.D.

757. COR. *The circle contains a maximum area within a given perimeter.*

EXERCISE 965. On the circumference of a given circle find the point such that the sum of the squares of its distances from two given points without the circle shall be a minimum.

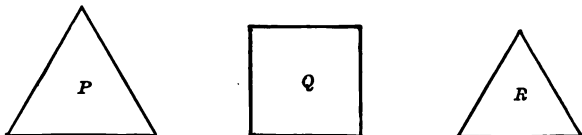
966. Of two given circles, one lies wholly within the other. Find the maximum and minimum chords of the outer that are tangent to the inner circle.

967. A line ABC is perpendicular to an indefinite line CM . Find the point P in CM at which the angle APB is a maximum.

968. Find a point within a quadrilateral such that the sum of the lines drawn from that point to the vertices shall be a minimum.

PROPOSITION VIII. THEOREM.

758. *Of two regular polygons having equal areas, that having the greater number of sides has the less perimeter.*



Given: P and Q , regular polygons of the same area, but Q with the greater number of sides;

To Prove: Perimeter of Q is less than perimeter of P .

Let R be a regular polygon having the same perimeter as Q and the same number of sides as P .

Then since $Q > R$, (756)

but $Q = P$, (Hyp.)

$P > R$;

\therefore the perimeter of $R <$ the perimeter of P ; (346)

\therefore the perimeter of $Q <$ the perimeter of P . Q.E.D.

759. COR. *The circumference of a circle is less than the perimeter of any polygon of equal area.*

EXERCISE 969. Through a given point within a given angle draw the intercept that cuts off the triangle of maximum area.

970. Through a point of intersection of two circles draw that intercept between the two circumferences which is a maximum.

971. In a given line find a point such that the sum of its distances from two given points without the line, and on the same side of it, shall be a minimum.

972. In a given line find the point such that the tangents drawn from it to a given circle contain the maximum angle.

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